Common Fixed Point Theorems in Uniform Spaces

Deepti Sharma

Department of Mathematics, Ujjain Engineering College, Ujjain (M.P.), 456010, India

***Abstract:* - In the process of generalization of metric spaces to Topological spaces, a few aspects of metric spaces are lost. Therefore, the requirement of generalization of metric spaces leads to the theory of uniform spaces. Uniform spaces stand somewhere in between metric spaces and general topological spaces. Khan[6] extended fixed point theorems due to Hardy and Rogers[2], Jungck[4] and Acharya[1] in uniform space by obtaining some results on common fixed points for a pair of commuting mappings defined on a sequentially complete Hausdorff uniform space. Rhoades et. al.[7] generalized the result of Khan[6] by establishing a general fixed point theorem for four compatible maps in uniform space .**

In this paper, a common fixed point theorem in uniform spaces is proved which generalizes the result of Khan[6] and Rhoades et al.[7] by employing the less restrictive condition of weak compatibility for one pair and the condition of compatibility for second pair, the result is proved for six self-mappings.

***AMS (2000) Subject Classification*:- 54H25, 47H10.**

***Keywords*: - Common fixed point, uniform space, compatible mappings and weakly compatible mappings.**

I. INTRODUCTION AND PRELIMINARY CONCEPTS

For the terminology, definition and basic properties of uniform spaces, the reader can refer to Joshi[3].

Following Khan[6] and Rhoades et. al.[7], we assume that throughout the paper, (X, **U**) stands for a sequentially complete Hausdorff uniform space and P be a fixed family of pseudo-metrics on X which generates the uniformity **U** . Following Kelley[5], they [6, 7] assumed :

(1.1) $V_{(p,\, r)} = \{ (x, y) : x, y \in X, p(x, y) < r \}$.

(1.2) G =

$$\left\{V : V = \bigcap_{i=1}^{n} V_{(p_i, r_i)} : p_i \in P, r_i > 0, i = 1, 2, \ldots, n\right\}$$

and for $\alpha > 0$,

(1.3) αV =

$$\left\{\bigcap_{i=1}^{n} V_{(p_i, \alpha r_i)} : p_i \in P, r_i > 0, i = 1, 2, \ldots, n\right\}.$$

Khan[6] and Rhoades et. al.[7] used the following well-known lemmas taken from Acharya[1] in order to prove their results.

Lemma 1.1. If $V \in G$ and $\alpha, \beta > 0$, then $\alpha(\beta V) = (\alpha\beta)V$.

Lemma 1.2. Let p be any pseudo-metric on X and $\alpha, \beta > 0$.

If $(x, y) \in \alpha V_{(p, r_1)} o \beta V_{(p, r_2)}$, then $p(x, y) < \alpha r_1 + \beta r_2$.

Lemma 1.3. If $x, y \in X$, then, for every V in G there is a positive number λ such that $(x, y) \in \lambda V$.

Lemma 1.4. For any arbitrary $V \in G$ there is a pseudo-metric p on X

such that $V = V_{(p,\, 1)}$. This p is called a Minkowski pseudo-metric of V.

Lemma 1.5. [6] Let $\{ y_n \}$ be a sequence in a complete metric space (X, p). If there exists $k \in (0, 1)$ such that $p(y_{n+1}, y_n) \le k\, p(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Definition 1.1 [7] Let A and B be two self-maps of a uniform space, p a pseudo-metric on X. A and B will be said to be compatible on X if $\lim_{n\to\infty} p(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\{Ax_n\}$ and $\{Bx_n\}$ converge to the same point t in X.

*Definition 1.2***.** Let A and B be self-mappings of a uniform space, p a pseudo-metric on X. Then the mappings A and B are said to be weakly compatible if they commute at their coincidence point, that is, Ax = Bx implies ABx = BAx for some $x \in X$.

II. MAIN RESULT

Khan[6] extended fixed point theorems due to Hardy and Rogers[2], Jungck[4] and Acharya[1] in uniform space by obtaining some results on common fixed points for a pair of commuting mappings defined on a sequentially complete Hausdorff uniform space**.** Rhoades et. al.[7] generalized the result of Khan[6] by establishing a general fixed point theorem for four compatible maps in uniform space . In this paper, a common fixed point theorem in uniform spaces is proved which generalizes the result of Khan[6] and Rhoades et al.[7] .

Theorem 2.1. Let A, B, S, T, P and Q be self-maps of X satisfying the following conditions:

(2.1) $(PQx, Ax) \in V_1$, $(STy, By) \in V_2$, $(PQx, By) \in V_3$ $(STy, Ax) \in V_4$,

$(PQx, STy) \in V_5$ implies that $(Ax, By) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ$

$\alpha_4 V_4 \circ \alpha_5 V_5$, where $\alpha_i = \alpha_i\ (x, y)$ are non-negative functions from $X \times X \to [0, 1)$ satisfying $\sup_{x,y \in X} \sum_{i=1}^{5} \alpha_i < 1$

and $\alpha_3 = \alpha_4$;

(2.2) $A(X) \subseteq ST(X)$, $B(X) \subseteq PQ(X)$;

(2.3) either A or PQ is continuous;

(2.4) (A, PQ) is compatible and (B, ST) is weakly compatible; and

(2.5) PQ = QP, ST = TS, AQ = QA and BT = TB,

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let $V \in G$ be arbitrary and p the Minkowski pseudo-metric of V. For $x, y \in X$, let $p(PQx, Ax) = r_1$, $p(STy, By) = r_2$, $p(PQx, By) = r_3$, $p(STy, Ax) = r_4$, $p(PQx, STy) = r_5$.

For any $\varepsilon > 0$, $(PQx, Ax) \in (r_1 + \varepsilon)V$, $(STy, By) \in (r_2 + \varepsilon)V$,

$(PQx, By) \in (r_3 + \varepsilon)V$, $(STy, Ax) \in (r_4 + \varepsilon)V$, $(PQx, STy) \in (r_5 + \varepsilon)V$.

From (2.1),

$(Ax, By) \in \alpha_1(r_1 + \varepsilon)V \circ \alpha_2(r_2 + \varepsilon)V \circ \alpha_3(r_3 + \varepsilon)V \circ \alpha_4(r_4 + \varepsilon)V \circ \alpha_5(r_5 + \varepsilon)V$.

where $\alpha_i = \alpha_i(x, y)$. Using Lemmas 1.1−1.3, we get

$p(Ax, By) < \alpha_1(r_1 + \varepsilon) + \alpha_2(r_2 + \varepsilon) + \alpha_3(r_3 + \varepsilon) + \alpha_4(r_4 + \varepsilon) + \alpha_5(r_5 + \varepsilon)$.

Since ε is arbitrary, we have

(2.6) $p(Ax, By) \le \alpha_1\, p(PQx, Ax) + \alpha_2\, p(STy, By)$

$+ \alpha_3\, p(PQx, By) + \alpha_4\, p(STy, Ax) + \alpha_5\, p(PQx, STy)$.

Now, let x_0 be an arbitary point in X. As $A(X) \subseteq ST(X)$ and $B(X) \subseteq PQ(X)$, then there exists $x_1, x_2 \in X$ such that $Ax_0 = STx_1 = y_0$ and $Bx_1 = PQx_2 = y_1$.In general construct a sequences $\{y_n\}$ in X such that $y_{2n} = STx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Bx_{2n+1} = PQx_{2n+2}$ for n = 0, 1, 2,… .Now, we show that $\{y_n\}$ is a Cauchy sequence in X. From (2.6), we have

$p(y_{2n}, y_{2n+1}) = p(Ax_{2n}, Bx_{2n+1})$

$\le \alpha_1\, p(PQx_{2n}, Ax_{2n}) + \alpha_2\, p(STx_{2n+1}, Bx_{2n+1})$

$+ \alpha_3\, p(PQx_{2n}, Bx_{2n+1}) + \alpha_4\, p(STx_{2n+1}, Ax_{2n})$

$+ \alpha_5\, p(PQx_{2n}, STx_{2n+1})$.

$$p(y_{2n}, y_{2n+1}) \le \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}\, p(y_{2n-1}, y_{2n}).$$

$$= \lambda\, p(y_{2n-1}, y_{2n}).$$

In general $p(y_n, y_{n+1}) \le \lambda\, p(y_{n-1}, y_n)$.From lemma (2.1), $\{y_n\}$ converges to some point z in X. Thus, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{STx_{2n+1}\}$ and $\{PQx_{2n+2}\}$ of sequence $\{y_n\}$ also converges to z in X.

Case I. Suppose A is continuous, we have $A^2x_{2n} \to Az$ and $A(PQ)x_{2n} \to Az$. The compatibility of the pair (A, PQ) gives that $(PQ)Ax_{2n} \to Az$.

Step 1. Putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (2.6), we have

$p(AAx_{2n}, Bx_{2n+1}) \le \alpha_1\, p(PQAx_{2n}, AAx_{2n})$

$+ \alpha_2\, p(STx_{2n+1}, Bx_{2n+1}) + \alpha_3\, p(PQAx_{2n}, Bx_{2n+1})$

$+ \alpha_4\, p(STx_{2n+1}, AAx_{2n}) + \alpha_5\, p(PQAx_{2n}, STx_{2n+1})$.

Letting $n \to \infty$ and using above results, we get

$p(Az, z) \le (\alpha_3 + \alpha_4 + \alpha_5)\, p(Az, z)$. So that Az = z.

Step 2. Since $A(X) \subseteq ST(X)$, there exists $u \in X$ such that z = Az = STu. Putting $x = x_{2n}$ and y = u in (2.6), we get

$p(Ax_{2n}, Bu) \le \alpha_1\, p(PQx_{2n}, Ax_{2n}) + \alpha_2\, p(STu, Bu)$

$+ \alpha_3\, p(PQx_{2n}, Bu) + \alpha_4\, p(STu, Ax_{2n}) + \alpha_5\, p(PQx_{2n}, STu)$.

Letting $n \to \infty$ and using above results, we get

$p(z, Bu) \le (\alpha_2 + \alpha_3)\, p(Bu, z)$. So that z = Bu. The weak compatibility of the pair (B, ST) gives that STBu = BSTu. Hence STz = Bz.

Step 3. Putting $x = x_{2n}$ and y = z in (2.6), we get

$p(Ax_{2n}, Bz) \leq \alpha_1 p(PQx_{2n}, Ax_{2n}) + \alpha_2 p(STz, Bz)$

$+ \alpha_3 p(PQx_{2n}, Bz) + \alpha_4 p(STz, Ax_{2n}) + \alpha_5 p(PQx_{2n}, STz)$.

Letting $n \to \infty$ and using above results, we get

$p(z, Bz) \leq (\alpha_3 + \alpha_4 + \alpha_5) p(Bz, z)$. So that $z = Bz = STz$.

Step 4. Putting $x = x_{2n}$ and $y = Tz$ in (2.6), we get

$p(Ax_{2n}, BTz) \leq \alpha_1 p(PQx_{2n}, Ax_{2n}) + \alpha_2 p(STTz, BTz)$

$+ \alpha_3 p(PQx_{2n}, BTz) + \alpha_4 p(STTz, Ax_{2n}) + \alpha_5 p(PQx_{2n}, STTz)$.

Since $BT = TB$ and $ST = TS$. We have $BTz = Tz$ and $ST(Tz) = Tz$.

Letting $n \to \infty$ and using above results, we get $z = Tz$. Now $STz = z$, which implies that $Sz = z$. Hence $Sz = Tz = Bz = z$.

Step 5. As $B(X) \subseteq PQ(X)$, there exists $v \in X$ such that $z = Bz = PQv$. Putting $x = v$ and $y = x_{2n+1}$ in (2.6), letting $n \to \infty$ and using above results, we get $p(Av, z) \leq (\alpha_1 + \alpha_4) p(Av, z)$. So that $Av = z = PQv$. As the pair (A, PQ) is compatible implies weakly compatible. Therefore $APQv = PQAv$ implies $Az = PQz$. Hence $PQz = Az = z$.

Step 6. Putting $x = Qz$ and $y = z$ in (2.6), As $AQ = QA$ and $PQ = QP$.We have $AQz = Qz$ and $PQ(Qz) = Qz$. Using above results, we get $p(Qz, z) \leq (\alpha_3 + \alpha_4 + \alpha_5) p(z, Qz)$.So that $Qz = z$. Therefore $PQz = z$, which implies that $Pz = z$.

Hence $Az = Bz = Sz = Tz = Pz = Qz = z$.

Thus, z is a common fixed point of A, B, S, T, P and Q.

Case II. Similarly by taking PQ is continuous, it can be proved that z is a common fixed point of A, B, S, T, P and Q

Uniqueness.

Let w is also a common fixed point of A, B, S, T, P and Q,

Then $w = Aw = Bw = Sw = Tw = Pw = Qw$.

Putting $x = z$ and $y = w$ in (2.6), we get $p(z, w) \leq (\alpha_3 + \alpha_4 + \alpha_5) p(z, w)$. So that $z = w$. Therefore, z is unique common fixed point of A, B, S, T, P and Q.

Corollary 2.1. Let A, B, S, T, P and Q be self-maps of X satisfying the conditions (2.1), (2.2), (2.3), (2.5) of theorem (2.1) and the pairs (A, PQ) and (B, ST) are compatible.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. As compatibility implies weak compatibility, the proof follows from theorem 2.1.

REFERENCES

[1]. Acharya, S.P.,Some Results on fixed points in uniform spaces, Yokohama Math. J. 22(1974), 105-116.
[2]. Hardy, G.E. and Rogers, T.D., A generalization of a fixed point theorem of Reich, Bull. Cand. Math. Soc., 16 (1973), 201-206.
[3]. Joshi, K.D., Introduction to General Topology, Wiley Eastern, 1992.
[4]. Jungck, G., Commuting mappings and fixed points, Amer. Math. Monthly, 83 (1976), 261-263.
[5]. Kelley, J. L., General topology, Van Nostrand, 1955.
[6]. Khan, M.S., Commuting maps and fixed points in uniform spaces, Bull. Cal. Math. Soc. 29(1981), 499-507.
[7]. Rhoades, B. E., Singh, B. and Chauhan, M.S., Fixed points of compatible mappings in uniform spaces, Ranchi Univ. Math. J., 25(1994), 83-87.