A Notes on Bailey's Transform

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***Abstract* — In this paper transformation formulae for poly-basic hypergeometric functions have been used by making use of Bailey's transform.**

***Keywords* — Hypergeometric functions, Summations, Transformation, Polybasic, Convergence.**

I. INTRODUCTION

The well-known Bailey's transformation states that, if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.1}$$

and $$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{n-r} v_{n+r} = \sum_{r=0}^{\infty} \delta_r u_r v_{r+2n} \tag{1.2}$$

where $\alpha_r, \delta_r, u_r, v_r$ are any functions of r only, and that the series for γ_n exists, then, subject to convergence,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.3}$$

Bailey's transform (1.3) has been exploited, both as a tool to find new transformations of both ordinary and basic, hypergeometric series and also to find new q-identities of Rogers-Ramanujam type. We shall make use of the following summation in our analysis.

$${}_3\Phi_3\begin{bmatrix} q^a & : & q_1^b & ; & zq^{a+1}q_1^{b+1} : q, q_1 : qq_1 : z \\ zq^{a+1} & : & zq_1^{b+1} ; & & zq^a q_1^b \end{bmatrix}_N$$

$$= \frac{(1-zq^a)(1-zq_1^b)}{(1-z)(1-zq^aq_1^b)} - \frac{[q^a;q]_{N+1}[q_1^b;q_1]_{N+1} z^{N+1}}{(1-z)(1-zq^aq_1^b)[zq^{a+1};q]_n[zq_1^{b+1};q_1]_N}. \tag{1.4}$$

As $N \to \infty$, yeilds

$${}_3\Phi_3\begin{bmatrix} q^a & : & q_1^b & ; & zq^{a+1}q_1^{b+1}; q, q_1; qq_1; z \\ zq^{a+1} & : & zq_1^{b+1} ; & & zq^a q_1^b \end{bmatrix}_N$$

$$= \frac{(1-zq^a)(1-zq_1^b)}{(1-z)(1-zq^aq_1^b)}. \tag{1.5}$$

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)[a;p]_k[c;q]_k c^{-k}}{(1-a)[q;q]_k[ap/c;p]_k} = \frac{[ap;p]_n[cq;q]_n}{[q;q]_n[ap/c;p]_n} \tag{1.6}$$

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;p]_k[c,a/bc;q]_k q^k}{(1-a)(1-b)[q,aq/b;q]_k[ap/c,bcp;p]_k} = \frac{[ap,bp;p]_n[cq,aq/bc;q]_n}{[q,aq/b;q]_n[ap/c,bcp;p]_n} \tag{1.7}$$

II. NOTATIONS AND DEFINITIONS

A basic hypergeometric series is generally defined to be a series of the type $\sum_{n=0}^{\infty} a_n z^n$ where a_{n+1}/a_n is a rational function of q^n, q being fixed complex parameters called the base of the series, usually with modulus less than one. An explicit representation of such series is given by:

$${}_r\Phi_s\begin{bmatrix} a_1, & a_2, \ldots, & a_r; q; z \\ b_1, & b_2, \ldots, & b_s; q^i \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} q^{i\binom{n}{2}} \frac{[a_1, a_2, \ldots, a_r; q]_n z^n}{[b_1, b_2, \ldots, b_s; q]_n} \tag{2.1}$$

where $\binom{n}{2} = n(n-1)/2$ And $[a_1, a_2, \ldots, a_r; q]_n = [a_1, q]_n [a_2, q]_n \ldots [a_r, q]_n$

With the q-shifted factorial defined by

$$[a;q]_n = \begin{cases} 1, & \text{if } n = 0 \\ (1-a)(a-aq)\ldots(1-aq^{n-1}), & \text{if } n = 1,2\ldots \end{cases} \tag{2.2}$$

For the convergence of the series (2.1) we need $|q| < 1$ and $|z| < \infty$ when $i = 1,2\ldots$ or max. $\{|q|, |z|\} < r$.When $i = 0$ provided no zeros appear in the denominator.

${}_r\Phi_s\begin{bmatrix} a_1, & a_2, \ldots, & a_r; q; z \\ b_1, & b_2, \ldots, & b_s; q^i \end{bmatrix}_N$ means that the series runs

From 0 to N only

The poly-basic hypergeometric series is defined as:

$$\Phi\begin{bmatrix} c_{1,1},\dots, & c_{1,r_1};\dots;c_{m,1},\dots, & c_{m,r_m};q_1,q_2,\dots,q_m;z \\ d_{1,1},\dots, & d_{1,s_1};\dots;d_{m,1},\dots, & d_{m,s_m} \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} z^n \prod_{j=1}^{m} \frac{\left[c_{j,1},\dots,c_{j,r_j};q_j\right]_n}{\left[d_{j,1},\dots,d_{j,s_j};q_j\right]_n} \tag{2.3}$$

III. MAIN RESULTS

Taking $u_r = v_r = 1$ and

$\delta_r = \frac{[q^a;q]_r\left[q_1^b;q_1\right]_r\left[xq^{a+1}q_q^{b+1};qq_1\right]_r x^r}{[xq^{a+1};q]_r\left[xq_1^{b+1};q_1\right]_r\left[xq^aq_1^b;qq_1\right]_r}$ in (1.2) we get

$$\gamma_n = \frac{[q^a;q]_n\left[q_1^b;q_1\right]_n\left[xq^{a+1}q_q^{b+1};qq_1\right]_n x^n(1-xq^{a+n})(1-xq_1^{b+n})}{[xq^{a+1};q]_n\left[xq_1^{b+1};q_1\right]_n\left[xq^aq_1^b;qq_1\right]_n(1-x)(1-xq^{a+n}q_1^{b+n})}$$

$$= \frac{(1-xq^a)(1-xq_1^b)(q^a;q)_n(q_1^b;q_1)_n x^n}{(x-1)(1-xq^aq_1^b)(xq^a;q)_n\left[xq_1^b;q_1\right]_n} \tag{3.1}$$

Putting these values in (1.1) (1.2) and (1.3) we get the new form of the Bailey's transform as

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r \tag{3.2}$$

Then

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)} \sum_{n=0}^{\infty} \frac{(q^a;q)_n\left[q_1^b;q_1\right]_n x^n}{(xq^a;q)_n\left[xq_1^b;q_1\right]_n}\alpha_n$$

$$= \sum_{n=0}^{\infty} \frac{(q^a;q)_n\left[q_1^b,q_1\right]_n\left[xq^{a+1}q_1^{b+1};qq_1\right]_n x^n}{(xq^{a+1};q)_n\left[xq^aq_1^b;qq_1\right]_n\left[xq_1^{b+1};q_1\right]_n}\beta_n \tag{3.3}$$

We shall make use of (3.2) and (3.3) in order to establish certain new transformation formulae

(a) Choosing

$\alpha_r = z^r$ in (3.2) we get

$$\beta_n = \frac{1-z^{n+1}}{1-z}$$

Putting these values in (3.3) we get,

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}$$

$${}_3\Phi_2\begin{bmatrix} q^a; & q_1^b;q,q_1;xz \\ xq^a; & xq_1^b \end{bmatrix}$$

$$= \frac{(1-xq^a)(1-xq_1^b)}{(1-z)(1-x)(1-xq^aq_1^b)} - \frac{z}{(1-z)}$$

$$\times\ {}_3\Phi_3\begin{bmatrix} q^a & : & q_1^b & ; & xq^{a+1};q_1^{b+1};q,q_1;qq_1;xz \\ xq^{a+1}: & & xq_1^{b+1}; & & xq^aq_1^b \end{bmatrix}, \tag{3.4}$$

(b) Next, taking

$$\alpha_r = \frac{[q_2^\alpha;q_2]_r\left[q_3^\beta;q_3\right]_r\left[zq_2^{\alpha+1}q_3^{\beta+1};q_2q_3\right]_r z^r}{[zq_2^{\alpha+1};q_2]_r\left[zq_3^{\beta+1};q_3\right]_r\left[zq_2^\alpha q_3^\beta;q_2q_3\right]_r}$$

In (3.2) and making use of (1.4) we get:

$$\beta_n = \frac{(1-zq_2^\alpha)(1-zq_3^\beta)}{(1-z)(1-zq_2^\alpha z_3^\beta)}$$

$$-\frac{[q_2^\alpha;q_2]_{n+1}\left[q_3^\beta;q_3\right]_{n+1} z^{n+1}}{(1-z)(1-zq_2^\alpha q_3^\beta)[zq_2^{\alpha+1};q_2]_n\left[zq_3^{\beta+1};q_3\right]_n} \tag{3.5}$$

Putting these values in (3.3) we have

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}.$$

$$\begin{bmatrix} q^a; & q_1^b & ; & q_2^\alpha,q_3^\beta;zq_2^{\alpha+1},zq_3^{\beta+1};q,q_1,q_2,q_3,q_2q_3;xz \\ xq^a, & xq_1^b; & & zq_2^{\alpha+1};zq_3^{\beta+1},zq_2^\alpha q_3^\beta \end{bmatrix}$$

$$= \frac{(1-zq_2^\alpha)(1-zq_3^\beta)(1-xq^a)(1-xq_1^b)}{(1-z)(1-x)(1-zq_2^\alpha q_3^\beta)(1-xq^aq_1^b)} - \frac{z(1-q_2^\alpha)(1-q_3^\beta)}{(1-z)(1-zq_2^\alpha q_3^\beta)}$$

$$\times\ {}_5\Phi_5\begin{bmatrix} q^a;q_1^b;xq^{a+1}q_1^{b+1};\ q_2^{\alpha+1};q_3^{\beta+1},q,q_1,qq_1,q_2,q_3;xz \\ xq^{a+1};\ xq_1^{b+1};xq^aq_1^b;zq_2^{\alpha+1};zq_3^{\beta+1} \end{bmatrix} \tag{3.6}$$

(c) Again, taking

$\alpha_r = \frac{(1-\alpha P^rQ^r)[\alpha;p]_r[\beta;Q]_r\beta^{-r}}{(1-\alpha)[Q;Q]_r[\alpha P/\beta;P]_r}$ in (3.2)

And making use of (1.6) we get,

$$\beta_n = \frac{[\alpha P;P]_n[\beta Q;Q]_n}{[Q;Q]_n[\alpha P/\beta;P]_n}$$

Putting these values in (3.3) we obtain,

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}\ {}_5\Phi_5\begin{bmatrix} q^a;q_1^b;\alpha PQ;\alpha;\beta;q;q_1,PQ,P,P,Q;x/\beta \\ xq^a;xq_1^b;\alpha;\alpha P/\beta;Q; \end{bmatrix}$$

$$\times {}_6\Phi_5 \begin{bmatrix} q^a; q_1^b; xq^{a+1}q_1^{b+1}; \alpha P; \beta Q; q, q_1, qq_1, P, Q; x \\ xq^{a+1}; xq_1^{b+1}; xq^a q_1^b; \alpha P/\beta; Q; \end{bmatrix}. \tag{3.7}$$

(d) Lastly, taking

$$\alpha_r = \frac{(1-\alpha P^r Q^r)(1-\beta P^r Q^{-r})[\alpha,\beta;P]_r[\gamma,\alpha/\beta\gamma;Q]_r Q^r}{(1-\alpha)(1-\beta)[Q,\alpha Q/\beta;Q]_r[\alpha P/\gamma,\beta\gamma P;P]_r}$$

In (3.2) and making use of (1.7) we get,

$$\beta_n = \frac{[\alpha P,\beta P;P]_n[\gamma Q,\alpha Q/\beta\gamma;Q]_n}{[Q,\alpha Q/\beta;Q]_n[\alpha P/\beta,\beta\gamma P;P]_n}$$

Now putting these values in (3.3) we find:

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^a q_1^b)}$$

$$\times {}_9\Phi_8 \begin{bmatrix} q^a; q_1^b; \alpha; \beta; \gamma, a/\beta\gamma; \alpha PQ; \beta P/Q; q, q_1, P, Q, PQ, P/Q; xQ \\ xq^a; xq_1^b; \alpha P/\gamma, \beta\gamma P, Q, \alpha Q/\beta; \alpha; \beta \end{bmatrix}$$

$$= {}_8\Phi_7 \begin{bmatrix} q^a; q_1^b; xq^{a+1}q_1^{b+1}; \alpha P, \beta P; \gamma Q, \alpha Q/\beta\gamma; q, q_1, qq_1, P, Q; x \\ xq^{a+1}; xq_1^{b+1}; xq^a q_1^b; \alpha P/\gamma, \beta\gamma P, Q, \alpha Q/\beta \end{bmatrix} \tag{3.8}$$

A number of other interesting results can also be obtained by taking suitable values of α_r and β_n.

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