Generating Function for Partitions with Parts in A.P

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***Abstract*: In this paper we derive the generating function for the number of ith over Ga partitions when the parts are in AP. we also obtain a formula for the number of the smallest parts of partitions of n.**

***Key Words*: Partition, r-partition, Over Partition, Gapartition, Generating function**

***Sub* : Classification :11P81 Elementry Theory of Partitions**

I. INTRODUCTION

Sylve.Corteel and Lovejoy [1] initiated the study of *overpartitions* of *n*. Hanuma Reddy.K [2] derived a formula for the i^{th} smallest parts of *overpartitions* of *n*. Ga partitions were introduced by G.V.R.K.Sagar[3]. In this paper we introduce, $j^{th}overGa\ partitions$ and $r-j^{th}overGa\ partitions$ of *n* and obtain generating functions in this context.

A partition of *n* is a $Ga\ partition$ if smallest parts are of the form $a^{k-1}, k \in N$. If $0 \le r \le n$, a $r - Ga\ partition\, of\ n$ is a $Ga\ partition\, of\ n$ with exactly *r* parts. A $(r-)\, j^{th}overGa\ partition\, of\ n$ is a $(r-)Ga\ partition\, of\ n$ in which first (equivalently, the final) occurrence of a part is over lined up to j times successively. We denote the set of $j^{th}overGa\ partitions$ of *n* by $\overline{Ga\,\xi(n)}^{\,j}$ and its cardinality by $\overline{Ga\ p(n)}^{\,j}$.

we also obtain the generating function for $\overline{Ga\,spt(n)}^{\,j}$.

II. GENERATING FUNCTION FOR NUMBER OF PARTITIONS WITH PARTS IN A.P.

Hanuma Reddy.K [2] established that the generating function for the number of partitions of n into r parts with s distinct parts $\mu_1, \mu_2, \mu_3, ..., \mu_s$, occurring with fixed frequencies $f_1, f_2, f_3, ..., f_s$ is

$$\sum_{n=1}^{\infty} p_r\left(\mu_1^{f_1}, \mu_2^{f_2}, ..., \mu_s^{f_s}; n\right) q^n = \frac{q^{f_1+(f_1+f_2)+(f_1+f_2+f_3)+...+(f_1+f_2+...+f_s)}}{\left(1-q^{f_1}\right)\left(1-q^{f_1+f_2}\right)\left(1-q^{f_1+f_2+f_3}\right)...\left(1-q^{f_1+f_2+...+f_s}\right)}$$

We extend this result as follows

2.1 Theorem : The generating function for the number of partitions of n into r parts with s distinct parts $\mu_1, \mu_2, \mu_3, ..., \mu_s$, occurring with fixed frequencies $f_1, f_2, f_3, ..., f_s$ and these parts belong to the set $S = \left\{a + (k-1)d \mid a, d, \mathrm{k} \in N\right\}$ is and are in A.P

$$\sum_{n=1}^{\infty} p_r\left(\mu_1^{f_1}, \mu_2^{f_2}, ..., \mu_s^{f_s}; n\right) q^n = \frac{q^{f_1.d+(f_1+f_2).d+(f_1+f_2+f_3).d+...+(f_1+f_2+...+f_{s-1}).d+(f_1+f_2+...+f_s).a}}{\left(1-q^{f_1.d}\right)\left(1-q^{(f_1+f_2).d}\right)\left(1-q^{(f_1+f_2+f_3).d}\right)...\left(1-q^{(f_1+f_2+...+f_s).d}\right)}$$

Proof: Any partition of a number n into r parts can be written in the form $\left(\mu_1^{f_1}, \mu_2^{f_2}, ..., \mu_s^{f_s}\right)$ where $\mu_1 > \mu_2 >, ..., \mu_s$, $f_1 + f_2 + ... + f_s = r$ and $\mu_i^{f_i}$ represents occurrence of μ_i, f_i times. A simple example is

$$\left([a+(s-1)d]^{f_1},[a+(s-2)d]^{f_2},.....(a+2d)^{s-2},(a+d)^{s-1},a^s\right).$$

Replacing s by $s+1$ in first part $a+(s-1)d$ there are two possibilities for the second part $a+(s-2)d$. We can either replace $a+(s-2)d$ by $a+(s-1)d$ or leave it unaltered.

Thus, given the "frequencies " $f_1, f_2, f_3,..., f_s$," systematic change of parts of all possible partitions into r parts of which s are distinct and occur with specified frequencies can be listed in an array and a generating function for each column can be defined.

Let $1\le s\le r, f_1, f_2,..., f_s$ be given frequencies. Starting with the r partition $\left([a+(s-1)d]^{f_1},[a+(s-2)d]^{f_2},...,(a+2d)^{s-2},(a+d)^{s-1},a^s\right)$ we consider the problem when the first place only is altered. Replace in the first part , s with (frequencies f_1) by $s+1$. The next part could be $a+(s-1)d$ or $a+(s-2)d$. These choices yield two partitions.

For example if we chose $a=3, d=2, s=2$ and $f_1=2, f_2=3$ our partition 5+5+3+3+3 gives two partitions 7+7+3+3+3 and 7+7+5+5+5.

We arrange them in two rows as follows:

5+5+3+3+3

7+7+3+3+3 and 7+7+5+5+5.

We repeat this process, now replacing 7 in the second row by 9, to get the third row 9+9+3+3+3, 9+9+5+5+5 and 9+9+7+7+7 Successive application of this process yields the following partitions in the $(j+1)^{th}$ row is, in the general case,

$$\{3+[(j+2)-1]2\}+\{3+[(j+2)-1]2\}+3+3+3, \{3+[(j+2)-1]2\}+\{3+[(j+2)-1]2\}+5+5+5,$$
$$...,\{3+[(j+2)-1]2\}+\{3+[(j+2)-1]2\}\{3+[(j+1)-1]2\}+\{3+[(j+1)-1]2\}\{3+[(j+2)-1]2\}$$

When the largest part is s with the frequency f_1, the $(j+1)^{th}$ row is

$$\left([a+(s+j-1)d]^{f_1},[a+(s-2)d]^{f_2},[a+(s-2)d]^{f_2},...,(a+2d)^{s-2},(a+d)^{s-1},a^s\right)$$

$$\left([a+(s+j-1)d]^{f_1},[a+(s-1)d]^{f_2},[a+(s-2)d]^{f_2},...,(a+2d)^{s-2},(a+d)^{s-1},a^s\right)$$

$$\left([a+(s+j-1)d]^{f_1},[a+(s)d]^{f_2},[a+(s-2)d]^{f_2},...,(a+2d)^{s-2},(a+d)^{s-1},a^s\right)$$

$$\left([a+(s+j-1)d]^{f_1},[a+(s+1)d]^{f_2},[a+(s-2)d]^{f_2},...,(a+2d)^{s-2},(a+d)^{s-1},a^s\right)$$

and so on and the last element in this row is

$$\left([a+(s+j-1)d]^{f_1},[a+(s+j-2)d]^{f_2},[a+(s-2)d]^{f_2},...,(a+2d)^{s-2},(a+d)^{s-1},a^s\right)$$

We repeat this process and obtain a triangular array in which the first part of the partitions in the first column increases by d with frequency f_1, the first and second parts increase by d with frequencies f_1 , f_2 and so on.

Let $\gamma = f_1.[a+(s-1)d]+f_2.[a+(s-2)d]+...+f_{s-2}.(a+2d)+f_{s-1}.(a+d)+f_s.a$ and $0<q<1.$ Associate

$q^{\gamma+\left[(j+k-1)d\right]f_1+(k-1)d.f_2}$ for the j^{th} element in the k^{th} column $\left(j\geq k\right)$.

This yields the required generating function in the following form

The generating function for each $\left(j^{th}\right)$ column is

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}q^{\gamma+\left[(j+k-1)d\right]f_1+(k-1)d.f_2}=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}q^{\gamma+(jd)f_1+(k-1)d(f_1+f)_2}=\sum_{k=0}^{\infty}\frac{q^{\gamma+(k-1)d(f_1+f)_2}}{\left(1-q^{d\,f_1}\right)}$$

The required generating function is $\sum_{k=0}^{\infty}\frac{q^{\gamma+(k-1)(f_1+f_2).d}}{\left(1-q^{d.f_1}\right)}=\frac{q^{\gamma}}{\left(1-q^{d.f_1}\right)\left(1-q^{d.(f_1+f_2)}\right)}$

which converges when $0<q<1.$

In the general case the generating function is

$$\frac{q^{\gamma}}{\left(1-q^{f_1d}\right)\left(1-q^{(f_1+f_2)d}\right)...\left(1-q^{(f_1+f_2+...+f_s)d}\right)}$$

where $\gamma=f_1.\left[a+(s-1)d\right]+f_2.\left[a+(s-2)d\right]+...+f_{s-2}.(a+2d)+f_{s-1}.(a+d)+f_s.a$

Hence the generating function for the number of partitions of n into r parts with s distinct parts $\mu_1,\mu_2,\mu_3,...,\mu_s$, occurring with fixed frequencies $f_1,f_2,f_3,...,f_s$ and these parts belong to the set $S=\left\{a+(n-1)d \mid a,d,n\in N\right\}$ is

$$\sum_{n=1}^{\infty}p_r\left(\mu_1^{f_1},\mu_2^{f_2},...,\mu_s^{f_s};n:S\right)q^n=\frac{q^{f_1.d+(f_1+f_2).d+(f_1+f_2+f_3).d+...+(f_1+f_2+...+f_{s-1}).d+(f_1+f_2+...+f_s).a}}{\left(1-q^{f_1.d}\right)\left(1-q^{(f_1+f_2).d}\right)\left(1-q^{(f_1+f_2+f_3).d}\right)...\left(1-q^{(f_1+f_2+...+f_s).d}\right)}$$

Note: If we put $a=1$ *and* $b=1$ in the above we get formula which is derived by Hanuma Reddy.K in [2]

2.2. $\overline{p_r(n)}^{\,j}=\frac{q^r(-j,q)_r}{(q)_r}$

Proof: Since the generating function for the number of partitions of n into r parts with s distinct parts $\mu_1,\mu_2,\mu_3,...,\mu_s$, occurring with fixed frequencies $f_1,f_2,f_3,...,f_s$ is

$$\sum_{n=1}^{\infty}p_r\left(\mu_1^{f_1},\mu_2^{f_2},...,\mu_s^{f_s};n\right)q^n=\frac{q^{f_1+(f_1+f_2)+(f_1+f_2+f_3)+...+(f_1+f_2+...+f_s)}}{\left(1-q^{f_1}\right)\left(1-q^{f_1+f_2}\right)\left(1-q^{f_1+f_2+f_3}\right)...\left(1-q^{f_1+f_2+...+f_s}\right)}$$

The generating function for the number of partitions of n into r parts for all linear combinations of $f_1,f_2,f_3,...,f_s$ such that $f_1+f_2+f_3+...+f_s=r$ is

$$\sum_{n=1}^{\infty}p_r(n)q^n=\frac{q^{f_1+(f_1+f_2)+(f_1+f_2+f_3)+...+(f_1+f_2+...+f_s)}}{\left(1-q^{f_1}\right)\left(1-q^{f_1+f_2}\right)\left(1-q^{f_1+f_2+f_3}\right)...\left(1-q^{f_1+f_2+...+f_s}\right)}$$

So the generating function for the number of j^{th} *overpartitions* of n into r parts

$$\sum_{n=1}^{\infty}\overline{p_r(n)}^{j}q^n=(j+1)^s\frac{q^{f_1+(f_1+f_2)+(f_1+f_2+f_3)+\ldots+(f_1+f_2+\ldots+f_s)}}{\left(1-q^{f_1}\right)\left(1-q^{f_1+f_2}\right)\left(1-q^{f_1+f_2+f_3}\right)\ldots\left(1-q^{f_1+f_2+\ldots+f_s}\right)}$$

$$=(j+1)^s\frac{q^r(-j,q)_r}{(q)_r}$$

and $\overline{p_r(n-a)}^{j}=\dfrac{q^{r+a}(-j,q)_r}{(q)_r}$

Examples

(i) $r=1$. Then $f_1=1$

$$\sum_{n=1}^{\infty}\overline{p_1(n)}^{j}q^n=(j+1)^1\frac{q^1}{\left(1-q^1\right)}=\frac{(j+1)q}{(1-q)}=\frac{q(-j,q)_1}{(q)_1}$$

(ii) $r=2$.

Then the possibilities are $s=1,\ 2$.

$\Rightarrow f_1=2$ or $f_1+f_2=1+1$

$$\sum_{n=1}^{\infty}\overline{p_2(n)}^{j}q^n=(j+1)^1\frac{q^2}{\left(1-q^2\right)}+(j+1)^2\frac{q^{1+(2)}}{\left(1-q^1\right)\left(1-q^2\right)}$$

$$=(j+1)\frac{q^2}{\left(1-q^2\right)}\left[1+(j+1)\frac{q}{(1-q)}\right]$$

$$=\frac{q^2(1+j)(1+jq)}{(1-q)\left(1-q^2\right)}=\frac{q^2(-j,q)_2}{(q)_2}$$

(iii) $r=3$

Then the possibilities are $s=1,\ 2,\ 3$.

$\Rightarrow f_1=3$ or $f_1+f_2=2+1$ or $f_1+f_2=1+2$ or $f_1+f_2+f_3=1+1+1$

$$\sum_{n=1}^{\infty}\overline{p_3(n)}^{j}q^n=(j+1)^1\frac{q^3}{\left(1-q^3\right)}+(j+1)^2\frac{q^{1+(1+2)}}{\left(1-q^1\right)\left(1-q^{1+2}\right)}+(j+1)^2\frac{q^{2+(2+1)}}{\left(1-q^2\right)\left(1-q^{2+1}\right)}$$
$$+(j+1)^3\frac{q^{1+(1+1)+(1+1+1)}}{\left(1-q^1\right)\left(1-q^{1+1}\right)\left(1-q^{1+1+1}\right)}$$

$$=\frac{q^3(1+j)(1+jq)\left(1+jq^2\right)}{(1-q)\left(1-q^2\right)\left(1-q^3\right)}=\frac{q^3(-j,q)_3}{(q)_3}$$

III. GENERATING FUNCTION FOR $\overline{Ga\,spt(n)}^{j}$.

In this section we propose a formula for finding the number of smallest parts of partitions of *n*.

3.1 Theorem: If $1\le a\le n$,

$\overline{Ga\,spt(n)}^{j} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(a^{k-1}, n-ta^{k-1}\right)}^{j} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(a^{k-1}+1, n-ta^{k-1}\right)}^{j} + (j+1)d(a,n)$ **Proof:** Let $\lambda = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}, \left(a^{k-1}\right)^{\alpha_l}\right)$ be any $r-Ga\ partition$ with l distinct parts $\mu_1, \mu_2, ..., \mu_{l-1}, \left(a^{k-1}\right)$.

For fixed t three cases arise.

Case1: Let $r > \alpha_l = t$. If $\left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}, \left(a^{k-1}\right)^{\alpha_l}\right) = (\lambda_1, \lambda_2, ..., \lambda_m)$, then $\lambda_{r-t} > a^{k-1}$.

By subtracting a^{k-1} from each part of $\left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}, \left(a^{k-1}\right)^{\alpha_l}\right)$, we get $n-ta^{k-1} = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}\right)$

Hence $n-ta^{k-1} = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t)-partition$ of $n-ta^{k-1}$ with $l-1$ distinct parts and each ³ $a^{k-1}+1$. Corresponding to this there are $(j+1)^{l-1}$ times $(r-t)-j^{th}overpartitions$ of $n-ta^{k-1}$. We know that the total number of $r-j^{th}overGa\ partitions$ is $(j+1)^{l}$. Thus the number of $r-j^{th}overpartitions$ having exactly t smallest parts each equal to a^{k-1} in the set $\left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}, \left(a^{k-1}\right)^{\alpha_l}\right)$ is $(j+1).\overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}^{j}$.

Case 2: $r > \alpha_l > t$. Then $\lambda_{r-t} = a^{k-1}$.

Omit a^{k-1}'s from last t places, then $n-ta^{k-1} = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}, \left(a^{k-1}\right)^{\alpha_l - t}\right)$

is a $(r-t)-Ga\ partition$ of $n-ta^{k-1}$ with l distinct parts, the least part being a^{k-1}. Corresponding to this, there are $(j+1)^{l}\left(r-j^{th}overGa\ partitions\right)$ of $n-ta^{k-1}$ with least part a^{k-1}.

Thus the number of $r-j^{th}overGa\ partitions$ having more than t smallest parts, each being a^{k-1} in $\left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}, \left(a^{k-1}\right)^{\alpha_l}\right)$ is $\overline{Ga\ f_{r-t}\left(a^{k-1}, n-ta^{k-1}\right)}^{j}$.

Case 3: $r = \alpha_l = t$. Then all parts of the Ga *partition* are equal and each part is of the form a^{k-1}. This Partition has $(j+1)$ times of $r-j^{th}overGa\ partitions$ of n.

The number of Ga *partitions* of n with equal parts each being a^{k-1} is equal to $d(a,n)$. The number of devises of N of the form a^{k-1} Since the number of such divisors of n is α, the number of $j^{th}overGa\ partitions$ of n is $(j+1)\alpha$.

From cases (1), (2) and (3) it follows that the numbe r of $r-j^{th}overGa\ partitions$ of n with smallest part a^{k-1} which occurs t times is

$$\overline{Ga\ f_{r-t}\left(a^{k-1}, n-ta^{k-1}\right)}^{j} + (j+1).\overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}^{j} + (j+1)d(a,n)$$

$$= \overline{Ga\ f_{r-t}\left(a^{k-1}, n-ta^{k-1}\right)}^{j} + \overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}^{j} + j.\overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}^{j} + (j+1)d(a,n)$$

$$= \overline{p_{r-t}\left(a^{k-1}, n-ta^{k-1}\right)}^{j} + j.\ \overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}^{j} + (j+1)d(a,n)$$

From [1] the number of smallest parts in j^{th}*overGa partitions* of n is

$$\overline{Ga\,spt\left(a^{k-1},n\right)}^{j} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(a^{k-1}, n-ta^{k-1}\right)}^{j} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p\left(a^{k-1}+1, n-ta^{k-1}\right)}^{j} + (j+1)\sum_{k=1}^{\infty} d(a,n)$$

3.2 Illustration

As an illustration we take $n = 6$, $a = 2$ and $j = 2$

The partitions under consideration are listed below with totals indicated at the end of the row. The total number of underlined parts is 171 as detailed below

$r=2$	$\lambda_1 = 5$	overptns	09
	$\lambda_1 = 4$		09
$r=3$	$\lambda_1 = 4$		18
	$\lambda_1 = 3$		24
	$\lambda_1 = 2$		12
$r=4$	$\lambda_1 = 3$		20
	$\lambda_1 = 2$		20
$r=5$	$\lambda_1 = 2$		34
	$\lambda_1 = 1$		25

We now apply the formula,

$$= \left[\left[\overline{p(1,5)}^{2} + \overline{p(1,4)}^{2} + \overline{p(1,3)}^{2} + \overline{p(1,2)}^{2} + \overline{p(1,1)}^{2}\right] + \left[\overline{p(2,4)}^{2} + \overline{p(2,2)}^{2}\right]\right]$$

$$+ 2\left\{\left(\overline{p(2,5)}^{2} + \overline{p(2,4)}^{2} + \overline{p(2,3)}^{2} + \overline{p(2,2)}^{2}\right) + \left(\overline{p(3,4)}^{2}\right)\right\} + 3.2$$

$$= \{(51+27+15+6+3)+(6+3)\} + 2\{(12+6+3+3)+(3)\} + 6$$

$$= 171$$

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