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On the Exponential Diophantine Equation $305^x + 503^y = z^2$

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Abstract: In this paper, we compute and prove the solution to the exponential Diophantine equation $305^x + 503^y = z^2$ where x, y and $z$are non-negative integers. The result indicate that the equation has no solution.

Keywords: divisibility; exponential Diophantine equation; modular arithmetic method; Quadratic residue; Prime number

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I.Introduction

An exponential Diophantine equation is a classical problem in Mathematics. Because one equation contains more than one unknown variable, the theory of number must be applied to find a solution. From 2006 to 2022, mathematicians investigated several exponential Diophantine equations of type $a^x + b^y = z^2$, where x, y and z are unknown variables and a, b are positive integers. The examples of the studied equation can be read in [1, 4, 6 -9, 11]. Recently, many researches on the exponential Diophantine equation have been published. For instance, N. Viriyapong and C. Viriyapong [12] studied the exponential Diophantine equation $255^x + 323^y = z^2$ in 2023. They proved that $(x, y, z) \in \{(1,0,16),(0,1,18)\}$ are two solutions to the equation. After that, S. Aggrawal et al. [2] proved that the exponential Diophantine equation $145^x + 85^y = z^2$ has a unique non-negative integer solution which is $(x, y, z) = (1,0,12)$. Next, another similar equation, $143^x + 485^y = z^2$, was studied[3]. In the same year, S. Tadee and N. Thaneepoon[10] studied the exponential Diophantine equation $6^x + p^y = z^2$ .They proved that if $p \le 7$ and x is even, then the solutions to the equation are $(p, x, y, z) \in \{(2,0,3,3),(3,0,1,2),(2,2,6,10),(3,4,6,45)\}$. In this paper, we use the knowledge in number theory to find the solution to the Diophantine equation $305^x + 503^y = z^2$ where x, y and z are non-negative integers.

II. Preliminaries

Theorem1: [5] The number 2 is a quadratic residue of primes of the form $p = 8k + 1$ and $p = 8k + 7$. The number 2 is not a quadratic residue of primes of the form $p = 8k + 3$ and $p = 8k + 5$.

Lemma2: If $z \in \mathbb{Z}$, then $z^2 \equiv 0,1,4 \pmod 5$.

Proof: Let $z \in \mathbb{Z}$. We have $z \equiv 0,1,2,3,4 \pmod 5$. It yields that $z^2 \equiv 0,1,4,9,16 \pmod 5 \equiv 0,1,4 \pmod 5$.

Lemma3: If y is an odd positive integer, then $3^y \equiv 2,3 \pmod 5$.

Proof: Suppose y is an odd positive integer. There exist $q \in \mathbb{Z}^+ \cup \{0\}$ such that $y = 4q+1$ or $4q+3$.

If $y = 4q + 1$, then we have $3^y = (3^4)^q 3 \equiv 3 (mod\ 5)$.

If $y = 4q + 3$, then we have $3^y = (3^4)^q 27$. Since $3^4 \equiv 1 (mod\ 5)$ and $27 \equiv 2 (mod\ 5)$, we obtain $3^y \equiv 1^q \cdot 2 (mod\ 5)$ or $3^y \equiv 2 (mod\ 5)$. Therefore, we can conclude that $3^y \equiv 2,3 (mod\ 5)$.

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III. Main Result

Main Theorem: The exponential Diophantine equation $305^x + 503^y = z^2$, where x, y and z are non-negative integers, has no solution.

Proof: Let x, y ,and $z \in \mathbb{Z}^+ \cup \{0\}$ such that

$$305^x + 503^y = z^2. \quad (1)$$

We separated into four cases as follows.

Case 1: $x = y = 0$. Equation (1)becomes $z^2 = 2$, which is impossible.

Case 2: $x = 0$ and $y > 0$. Equation (1) becomes $z^2 = 1 + 503^y$, which implies that $z^2 \equiv 2 \pmod{251}$. This contradicts Theorem1 because of $251 \equiv 3 \pmod 8$.

Case 3: $x > 0$ and $y = 0$. Equation (1) becomes $z^2 = 305^x + 1$, which results in $z^2 \equiv 2 \pmod 4$. This is impossible because of $z^2 \equiv 0,1 \pmod 4$.

Case 4: $x > 0$ and $y > 0$. To clarify, we consider y to be two sub cases as follows.

Sub case 4.1: y is a positive odd integer. By Lemma3, we have $3^y \equiv 2,3 \pmod 5$, while equation (1) yields $z^2 \equiv 3^y \pmod 5$. Thus, we have $z^2 \equiv 2,3 \pmod 5$, which contradicts Lemma 2.

Subcase 4.2: y is a positive even integer. We have $3^y \equiv 1 \pmod 4$.Equation (1) yields $z^2 \equiv 1 + 3^y \pmod 4$ or $z^2 \equiv 2 \pmod 4$, which contradicts $z^2 \equiv 0,1 \pmod 4$.

IV. Conclusion

We studied solution to the exponential Diophantine equation $305^x + 503^y = z^2$ where x, y and z are non-negative integers. We derive two Lemmas to prove that the equation has no solution**.**

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References

1. Acu, D., (2007) On a Diophantine Equation $2^x + 5^y = z^2$, General Mathematics, 15(4), 145-148.
2. Aggarwal, S., Swarup, C., Gupta, D., and Kumar, S., (2023) Solution of the Diophantine Equation $143^x + 85^y = z^2$, International Journal of Progressive Research in Science and Engineering, 4(22), 5 – 7.
3. Aggarwal, S., Kumar, S., Gupta, D., and Kumar, S., (2023) Solution of the Diophantine Equation $143^x + 485^y = z^2$, International Research Journal of Modernization in Engineering Technology and Science, 5(2), 555 – 558.
4. Behera, S. P.and Panda, A.C., (2021) Nature of the Diophantine Equation $4^x + 12^y = z^2$, International Journal of Innovative Research in Computer Science & Technology (IJIRCST), 9(6), 11-12.
5. Burton, D. M., (2011) Elementary Number Theory, The McGraw-Hill and in positive integers Companies.
6. Kumar, S. and Aggarwal, S., (2021) On the Exponential Diophantine Equation $439^p + 457^q = r^2$, Journal of Emerging Technologies and Innovation Research, 8(3), 2357-2361.
7. Sroysang, B., (2013) More on the Diophantine Equation $2^x + 19^y = z^2$, International Journal of Pure and Applied Mathematics, 88(1), 157-160.

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue II, February 2024

8. Sugandha, A., Tripena, A., Prabowo, A. and Sukono, F., (2018) Nonlinear Diophantine Equation $11^x + 13^y = z^2$, IOP Conf. Series: Materials Science and Engineering, 332, 1- 4.
9. Suvarnamani, A., (2011) On two Diophantine Equation $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, Science and Technology RMUTT Journal, 1(1), 25-28.
10. Tadee, S., and Thaneepoon, N., (2023) On the Diophantine equation $6^x + p^y = z^2$where p is Prime, International Journal of Mathematics and Computer Science, 18(4), 737-741.
11. Thongnak, S., Chuayjan, W. and Kaewong, T., (2022) On the Exponential Diophantine Equation $2^x + 15^y = z^2$, Annals of Pure and Applied Mathematics, 26(1), 1-5.
12. Viriyapong, N. and Viriyapong, C., (2023) On the Diophantine equation $255^x + 323^y = z^2$, International Journal of Mathematics and Computer Science, 18(3), 521 – 523.