Inverse Transient Thermoelastic Problem of Semi-Infinite Rectangular Plate

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Abstract- This paper is concerned with transient thermoelastic problem in which we need to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of semi-infinite rectangular plate when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.

Key Words: Semi-infinite rectangular plate, transient problem, Integral transform, inverse problem

I. INTRODUCTION


In this paper, an attempt has been made to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of semi-infinite rectangular plate occupying the space D: 0 ≤ x ≤ a, 0 ≤ y ≤ ∞ with the boundary conditions that the temperature is maintained at zero on the edges y = 0, ∞ and on the edge x = a of a thin rectangular plate respectively.

II. STATEMENT OF THE PROBLEM

Consider semi-infinite rectangular plate occupying the space D: 0 ≤ x ≤ a, 0 ≤ y ≤ ∞. The displacement components u_x and u_y in the x and y-direction represented in the internal form as [2] are

\[ u_x = \int \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \]  \hspace{1cm} (2.1)

\[ u_y = \int \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \] \hspace{1cm} (2.2)

v and α are the Poisson’s ratio and the linear coefficient of thermal expansion of the material of the plate respectively and U(x,y, t) is the Airy’s stress function which satisfy the following relation

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) U = -\alpha E \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T \] \hspace{1cm} (2.3)

where E is the Young’s modulus of elasticity and T is the temperature of the plate satisfying the differential equation

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{k} \frac{\partial T}{\partial t} \] \hspace{1cm} (2.4)

subject to the initial condition

\[ T(x, y, 0) = 0 \] \hspace{1cm} (2.5)

the boundary conditions

\[ T(0, y, t) = 0 \] \hspace{1cm} (2.6)

\[ T(a, y, t) = g(y, t) \text{ (unknown)} \] \hspace{1cm} (2.7)
\[ T(x,0,t) = 0 \]  
(2.8)

\[ T(x,\infty,t) = 0 \]  
(2.9)

the interior condition

\[ T(\xi,y,t) = f(y,t), \ 0 < \xi < a \quad \text{(known)} \]  
(2.10)

where \( k \) is the thermal diffusivity of the material of the plate.

The stress components in terms of \( U \) are given by

\[ \sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \]  
(2.11)

\[ \sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \]  
(2.12)

\[ \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \]  
(2.13)

Equations (2.1) to (2.13) constitute the mathematical formulation of the problem under consideration.

### III. SOLUTION OF THE PROBLEM

Applying Fourier sine transform to the equations (2.4), (2.5), (2.6), (2.7) and (2.10) and using (2.8), (2.9) one obtains

\[ \frac{d^2 \tilde{T}_s}{dx^2} - p^2 \tilde{T}_s = \frac{1}{k} \frac{dT}{dt} \]  
(3.1)

where \( p^2 = m^2 \pi^2 \)

\[ \tilde{T}_s(x,m,0) = 0 \]  
(3.2)

\[ \tilde{T}_s(0,m,t) = 0 \]  
(3.3)

\[ \tilde{T}_s(a,m,t) = g_s(m,t) \]  
(3.4)

\[ \tilde{T}_s(\xi,m,t) = \tilde{g}_s(m,t) \]  
(3.5)

\[ \tilde{T}_s(\xi,m,t) = f_s(m,t) \]  
(3.6)

where \( \tilde{T}_s \) denotes Fourier sine transform of \( T \) and \( m \) is sine transform parameter.

Applying Laplace transform to the equations (3.1), (3.4), (3.5), (3.6) and using (3.3) one obtains

\[ \frac{d^2 \tilde{T}_s}{dx^2} - q^2 \tilde{T}_s = 0 \]  
(3.7)

where \( q^2 = p^2 + \frac{s}{k} \)  
(3.8)

\[ \hat{T}_s^*(0,m,s) = 0 \]  
(3.9)

\[ \hat{T}_s^*(a,m,s) = \hat{g}_s^*(m,s) \]  
(3.10)

\[ \hat{T}_s^*(\xi,m,s) = \hat{f}_s^*(m,s) \]  
(3.11)

where \( \hat{T}_s^* \) denotes Laplace transform of \( T \) and \( s \) is Laplace transform parameter.

Equation (3.6) is a second order differential equation whose solution gives

\[ \hat{T}_s^*(x,m,s) = A e^{\xi x} + B e^{-\xi x} \]  
(3.12)

where \( A, B \) are arbitrary constants.

Using (3.9) and (3.11) in (3.12) one obtains

\[ A + B = 0 \]  
(3.13)

\[ A e^{\xi a} + B e^{-\xi a} = \hat{f}_s^*(m,s) \]  
(3.14)

Solving (3.13) and (3.14) one obtains

\[ A = \frac{\hat{f}_s^*(m,s)}{e^{\xi a} - e^{-\xi a}}, \quad B = -\frac{\hat{f}_s^*(m,s)}{e^{\xi a} - e^{-\xi a}} \]

Substituting the values of \( A \) and \( B \) in (3.12) one obtains

\[ \hat{T}_s^*(x,m,s) = \hat{f}_s^*(m,s) \frac{\sinh(qx)}{\sinh(q\xi)} \]  
(3.15)

Using the condition (3.10) to the solution (3.15) one obtains

\[ \hat{g}_s^*(m,s) = \hat{f}_s^*(m,s) \frac{\sinh(qa)}{\sinh(q\xi)} \]  
(3.16)

Applying inverse Laplace transform to the equation (3.15) one obtains

\[ \hat{T}_s(x,m,t) = L^{-1} \left[ \hat{f}_s^*(m,s) \frac{\sinh(qx)}{\sinh(q\xi)} \right] \]  
(3.17)

To evaluate \( L^{-1} \left[ \hat{f}_s^*(m,s) \right] \)

\[ \hat{g}_1(s) = \frac{\sinh(qx)}{\sinh(q\xi)} \]  
(3.18)

Using inversion integral to the equation (3.17) one obtains
\[- \frac{g_1(t)}{2\pi} = \frac{1}{e^{iqs} - e^{q\xi}} \sum_{n=1}^{\infty} \sinh(q) \sinh(q) \int_{2\pi} e^{s} \frac{\sinh(qs)}{\sinh(q\xi)} ds \quad (3.19)\]

Now to calculate the inversion integral (3.19) where c is greater than the real part of all singularities of the integrand. The integral is a single valued function of s in the region bounded by the closed Bromwich contour of the figure given below:

The line NL is chosen so as to lie all the poles to the right, which are given by

\[s = s_n = \frac{\pi}{\xi}, \quad n = 1, 2, \ldots\]

Choosing the contour so that the curved portion LMN is an arc of the circle \(\Gamma\) with center at the origin and radius \(R = n(1+1/2)(\pi/\xi)^2\), so that it will not pass through zero of \(\sinh(q)\).

The integral over the circular arc tends to zero as \(n \to \infty\).

Now \(\sinh(q) = 0\) gives \(q = (\im\pi/\xi)\) i.e. \(s = s_n = (\im\pi/\xi), \quad n = 1, 2, 3, \ldots\)

Therefore

Residue at \(s_n = \im\pi/\xi\)

\[\lim_{s \to s_n} \left( s-s_n \right) e^{s} \frac{\sinh(qs)}{\sinh(qs)} (x, \frac{\sqrt{s}+p^2}{\xi}) \]

\[\lim_{s \to s_n} \left( s-s_n \right) e^{s} \frac{\sinh(qs)}{\sinh(qs)} (x, \frac{\sqrt{s}+p^2}{\xi}) \]

\[\lim_{s \to s_n} \left( s-s_n \right) e^{s} \frac{\sinh(qs)}{\sinh(qs)} (x, \frac{\sqrt{s}+p^2}{\xi}) \]

\[= \frac{2k\pi}{\xi} \cos n\pi \left( \frac{1}{\xi} \right) \sin \left( \frac{\pi}{\xi} \right) \frac{\sinh(qs)}{\sinh(qs)} (x, \frac{\sqrt{s}+p^2}{\xi}) \]

Hence the value of \(\frac{\im g_1(t)}{2\pi}\) is given by

\[\frac{g_1(t)}{2\pi} = \frac{2k\pi}{\xi} \sum_{n=1}^{\infty} (-1)^{n+1} \sin \left( \frac{\pi}{\xi} \right) \frac{\sinh(qs)}{\sinh(qs)} (x, \frac{\sqrt{s}+p^2}{\xi}) \]

Applying the Convolution Theorem to the equation (3.17) one obtains

\[\overline{T}_{s}(x,m,t) = \frac{2k\pi}{\xi} \sum_{n=1}^{\infty} (-1)^{n+1} \sin \left( \frac{\pi}{\xi} \right) x \]

\[\times \int_{0}^{t} \overline{f}_{s}(m,t)e^{-k\left( \frac{p^2-n^2\pi^2}{\xi^2} \right)(t-t')} \]

Also by using the result (3.19), the equation (3.16) gives

\[\frac{\im g_1(t)}{2\pi} = \frac{2k\pi}{\xi} \sum_{n=1}^{\infty} (-1)^{n+1} \sin \left( \frac{\pi}{\xi} \right) x \]

\[\times \int_{0}^{t} \overline{f}_{s}(m,t)e^{-k\left( \frac{p^2-n^2\pi^2}{\xi^2} \right)(t-t')} \]

Applying inverse Fourier sine transform to the equations (3.20) and (3.21) one obtain the expressions for the temperature distribution \(T(x, y, t)\) and unknown temperature gradient \(g(y, t)\) as

\[T(x, y, t) = \frac{2k\pi}{\xi} \sum_{n=1}^{\infty} \sin \left( \frac{\pi}{\xi} \right) x \]

\[\times \int_{0}^{t} \overline{f}_{s}(m,t)e^{-k\left( \frac{p^2-n^2\pi^2}{\xi^2} \right)(t-t')} \]

\[g(y, t) = \frac{2k\pi}{\xi} \sum_{n=1}^{\infty} \sin \left( \frac{\pi}{\xi} \right) x \]

\[\times \int_{0}^{t} \overline{f}_{s}(m,t)e^{-k\left( \frac{p^2-n^2\pi^2}{\xi^2} \right)(t-t')} \]

where \(\overline{f}_{s}(m,t) = \int_{0}^{\infty} f(y, t) \sin py dy \)

Substituting the value of \(T(x,y, t)\) from (3.22) in (2.3) one obtains the expression for Airy’s stress function \(U(x,y, t)\) as

\[U(x,y, t) = -\frac{\alpha E}{p^2} \left( \frac{2k\pi}{\xi} \right) \sum_{n=1}^{\infty} \sin \left( \frac{\pi}{\xi} \right) x \]

\[\times \int_{0}^{t} \overline{f}_{s}(m,t)e^{-k\left( \frac{p^2-n^2\pi^2}{\xi^2} \right)(t-t')} \]
\[ \times \int_0^t f_s (m, t') e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \quad (3.24) \]

**IV. DETERMINATION OF THERMOELASTIC DISPLACEMENT**

Substituting the value of \( U(x, y, t) \) from (3.24) in (2.1) and (2.2) one obtains the thermoelastic displacement functions \( u_x \) and \( u_y \) as

\[
\begin{align*}
\quad u_x &= \left( \frac{2ak\pi}{\xi^3} \right) \sum_{m=1}^{\infty} \sin py \sum_{n=1}^{\infty} (-1)^{n+1} n^2 \sin \left( \frac{n\pi}{\xi} \right) x \\
&\times \left[ 2 - \frac{\nu n^2}{\xi^2 p^2} \right] \\
\times \int_0^t f_s (m, t') e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \quad (4.1) \\
\quad u_y &= \left( \frac{2ak}{p^2 \xi^2} \right) \sum_{m=1}^{\infty} \cos py \sum_{n=1}^{\infty} (-1)^{n+1} n \sin \left( \frac{n\pi}{\xi} \right) x \\
&\times \left[ \frac{\pi^2 n^2}{\xi p^2} - \nu + 1 \right] \\
\times \int_0^t f_s (m, t') e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \quad (4.2) \\
\end{align*}
\]

**V. DETERMINATION OF STRESS FUNCTIONS**

Using (3.24) in (2.11), (2.12) and (2.13) the stress functions are obtained as

\[
\begin{align*}
\quad \sigma_{xx} &= \alpha E \left( \frac{2k}{\xi^2} \right) \sum_{m=1}^{\infty} \sin py \sum_{n=1}^{\infty} (-1)^{n+1} n \sin \left( \frac{n\pi}{\xi} \right) x \\
&\times \int_0^t f_s (m, t') e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \\
\quad \sigma_{yy} &= \alpha E \left( \frac{2k \pi^2}{p^2 \xi^2} \right) \sum_{m=1}^{\infty} \sin py \sum_{n=1}^{\infty} (-1)^{n+1} n^3 \sin \left( \frac{n\pi}{\xi} \right) x \\
&\times \int_0^t f_s (m, t') e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \\
\end{align*}
\]

\[
\begin{align*}
\quad \sigma_{xy} &= \frac{\alpha E}{p} \left( \frac{2k \pi}{\xi^2} \right) \sum_{m=1}^{\infty} \cos py \sum_{n=1}^{\infty} (-1)^{n+1} n^2 \cos \left( \frac{n\pi}{\xi} \right) x \\
&\times \int_0^t f_s (m, t') e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \quad (5.2) \\
\end{align*}
\]

\[
\begin{align*}
\quad \sigma_{xx} &= \frac{\alpha E}{p} \left( \frac{2k \pi^2}{p \xi^2} \right) \sum_{m=1}^{\infty} \cos py \sum_{n=1}^{\infty} (-1)^{n+1} n \sin \left( \frac{n\pi}{\xi} \right) x \\
&\times \int_0^t f_s (m, t') e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \\
\end{align*}
\]

**VI. SPECIAL CASE**

Set \( f(y,t) = \left( \frac{(1-e^{-t})y\xi}{1 + y^2} \right) \)

Applying Fourier sine transform to the equation (6.1) one obtains

\[
\begin{align*}
\bar{f}_s (m,t) &= \int_0^\infty \left( \frac{(1-e^{-t})y\xi}{1 + y^2} \right) \sin py \, dy \\
&= \left( 1 - e^{-t} \right) \left( \frac{\pi^2}{2} e^{-p} \right) \quad (6.2) \\
\end{align*}
\]

Substituting the value of \( \bar{f}_s (m,t) \) from (6.2) in the equations (3.22), (3.23), one obtains

\[
\begin{align*}
\quad T(x,y,t) &= \left( \frac{\pi k}{\xi} \right) \sum_{m=1}^{\infty} e^{-p} \sin py \sum_{n=1}^{\infty} (-1)^{n+1} n \sin \left( \frac{n\pi}{\xi} \right) x \\
&\times \int_0^t \left( 1 - e^{-t} \right) e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \quad (6.3) \\
\end{align*}
\]

\[
\begin{align*}
\quad g(y,t) &= \left( \frac{\pi k}{\xi} \right) \sum_{m=1}^{\infty} e^{-p} \sin py \sum_{n=1}^{\infty} (-1)^{n+1} n \sin \left( \frac{n\pi}{\xi} \right) a \\
&\times \int_0^t \left( 1 - e^{-t} \right) e^{-k \left( p^2 + n^2 \frac{\pi^2}{\xi^2} \right) (t-t')} \, dt' \quad (6.4) \\
\end{align*}
\]
VII. NUMERICAL RESULT

Set \( \beta = \frac{\pi k}{\xi} \), \( \pi = 3.14 \), \( a = 2 \) m, \( \xi = 1.5 \) m, \( t = 1 \) sec. and \( k = 0.86 \) in the equation (6.4) to obtain

\[
g(y,t) = \sum_{m=1}^{\infty} e^{-\beta m^2} \sin \pi y \sum_{n=1}^{\infty} (-1)^{n+1} n \sin (4.2n) \\
\times \int_{0}^{1} (1-e^{-t'}) e^{-0.86(2.47m^2+4.4n^2)(1-t')} dt'
\]

(7.1)

VIII. CONCLUSION

In both the problems, the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of semi-infinite rectangular beam have been investigated with the aid of integral transform techniques. The expressions are obtained in terms of Bessel’s function in the form of infinite series. The results that are obtained can be applied to the design of useful structures or machines in engineering applications.

Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions

REFERENCES