Solution of Linear Partial Integro-Differential Equations using Kamal Transform

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Abstract: In this paper, we used Kamal transform for solving linear partial integro-differential equations. The technique is described and illustrated with application. This technique gives the exact results using very less computational work.

Keywords: Linear partial integro-differential equation, Kamal transform, Convolution theorem, Inverse Kamal transform.

I. INTRODUCTION


The general linear partial integro-differential equation is given by

\[ \sum_{i=0}^{m} a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu 
+ \sum_{i=0}^{r} d_i \int_{0}^{t} k_i (t,s) \frac{\partial^i u(x,s)}{\partial x^i} + f(x,t) = 0 \ldots \ldots \ldots (1) \]

(with prescribed conditions), where the kernels \( k_i(t,s) \) and \( f(x,t) \) are known functions and \( a_i, b_i, c \) and \( d_i \) are constants or functions of \( x \).

The Kamal transform of the function \( F(t) \) is defined as [6]:

\[ K\{F(t)\} = \int_{0}^{\infty} F(t)e^{-at} \, dt \]

\[ = G(v), t \geq 0, k_1 \leq v \leq k_2 \ldots \ldots \ldots (2) \]

Where \( K \) is Kamal transform operator.

The Kamal transform of the function \( F(t) \) exist if \( F(t) \) is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Kamal transform of the function \( F(t) \).


The object of the present study is to determine exact solutions for linear partial integro-differential equations using Mahgoub transform without large computational work.

II. LINEARITY PROPERTY OF KAMAL TRANSFORMS

\[ K\{aF(t) + bG(t)\} = aK\{F(t)\} + bK\{G(t)\} \]

Where \( a, b \) are arbitrary constants.
III. KAMAL TRANSFORM OF SOME ELEMENTARY FUNCTIONS [6, 8]:

<table>
<thead>
<tr>
<th>S.N.</th>
<th>(F(t))</th>
<th>(K{F(t)} = G(v))</th>
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</thead>
<tbody>
<tr>
<td>1.</td>
<td>1</td>
<td>(v)</td>
</tr>
<tr>
<td>2.</td>
<td>(t)</td>
<td>(v^2)</td>
</tr>
<tr>
<td>3.</td>
<td>(t^2)</td>
<td>(2! v^3)</td>
</tr>
<tr>
<td>4.</td>
<td>(t^n, n \in N)</td>
<td>(n! v^{n+1})</td>
</tr>
<tr>
<td>5.</td>
<td>(e^{at})</td>
<td>(\frac{v}{1 - av})</td>
</tr>
<tr>
<td>6.</td>
<td>(\sin at)</td>
<td>(\frac{av^2}{1 + a^2 v^2})</td>
</tr>
<tr>
<td>7.</td>
<td>(\cos at)</td>
<td>(\frac{v}{1 + a^2 v^2})</td>
</tr>
<tr>
<td>8.</td>
<td>(\sinh t)</td>
<td>(\frac{av^2}{1 - a^2 v^2})</td>
</tr>
<tr>
<td>9.</td>
<td>(\cosh t)</td>
<td>(\frac{v}{1 - a^2 v^2})</td>
</tr>
</tbody>
</table>

IV. KAMAL TRANSFORM OF SOME PARTIAL DERIVATIVES OF THE FUNCTION \(u(x, t)\) [7]:

If \(K\{u(x, t)\} = G(x, v)\) then

(a) \(K \left\{ \frac{\partial u(x, t)}{\partial t} \right\} = \frac{1}{v} G(x, v) - u(x, 0) \ldots (3)\)

(b) \(K \left\{ \frac{\partial^2 u(x, t)}{\partial t^2} \right\} = \frac{1}{v^2} G(x, v) - \frac{1}{v} u(x, 0)\)

(c) \(K \left\{ \frac{\partial^n u(x, t)}{\partial t^n} \right\} = \frac{1}{v^n} G(x, v) - \frac{1}{v^{n-1}} u(x, 0)\)

\[- \frac{1}{v^n - 2} u_t(x, 0) - \ldots - u_{t(n-1)}(x, 0) \ldots (5)\]

(d) \(K \left\{ \frac{\partial u(x, t)}{\partial x} \right\} = \frac{dG(x, v)}{dx} \ldots \ldots \ldots (6)\)

(e) \(K \left\{ \frac{\partial^2 u(x, t)}{\partial x^2} \right\} = \frac{d^2G(x, v)}{dx^2} \ldots \ldots \ldots (7)\)

(f) \(K \left\{ \frac{\partial^n u(x, t)}{\partial x^n} \right\} = \frac{d^n G(x, v)}{dx^n} \ldots \ldots \ldots (8)\)

V. CONVOLUTION OF TWO FUNCTIONS [14]:

Convolution of two functions \(F(t)\) and \(H(t)\) is denoted by \(F(t) \ast H(t)\) and it is defined by

\[F(t) \ast H(t) = F \ast H = \int_0^t F(x)H(t - x)\, dx\]

\[= \int_0^t H(x)F(t - x)\, dx\]

VI. CONVOLUTION THEOREM FOR KAMAL TRANSFORMS [8]:

If \(K\{F(t)\} = G(v)\) and \(K\{H(t)\} = I(v)\) then

\[K\{F(t) \ast H(t)\} = K\{F(t)\}K\{H(t)\} = G(v)I(v)\]

VII. INVERSE KAMAL TRANSFORMS

If \(K\{F(t)\} = G(v)\) then \(F(t)\) is called the inverse Kamal transform of \(G(v)\) and mathematically it is defined as

\[F(t) = K^{-1}\{G(v)\}\]

Where \(K^{-1}\) is the inverse Kamal transform operator.

VIII. INVERSE KAMAL TRANSFORM OF SOME ELEMENTARY FUNCTIONS

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<td>(\frac{t^2}{2!})</td>
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IX. KAMAL TRANSFORM FOR LINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS:

In this section, we present Kamal transform for solving linear partial integro-differential equations given by (1). In this work, we will assume that the kernels \(k_i(t, s)\) of (1) are difference kernel that can be expressed by difference \((t - s)\). The linear partial integro-differential equation (1) can thus be expressed as

\[
\sum_{i=0}^{m} a_i \frac{\partial^i u(x, t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x, t)}{\partial x^i} + cu + \sum_{i=0}^{m} d_i \int_0^t k_i(t - s) \frac{\partial^i u(x, s)}{\partial x^i} + f(x, t)
\]

\[= 0 \ldots \ldots \ldots (9)\]
Applying the Kamal transform to both sides of (9), we have
\[ \sum_{i=0}^{m} a_i K \left\{ \frac{\partial^i u(x, t)}{\partial t^i} \right\} + \sum_{r=0}^{n} b_r K \left\{ \frac{\partial^r u(x, t)}{\partial x^r} \right\} + cK[u] \]
\[ = \sum_{i=0}^{m} a_i \int_0^t k_i (t-s) \frac{\partial^i u(x,s)}{\partial x^i} ds + K\{f(x,t)\} = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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