Mahgoub Transform of Bessel's Functions

Sudhanshu Aggarwal[1], Nidhi Sharma[2], Raman Chauhan[3]
[1]*Assistant Professor, Department of Mathematics, National P.G. College Barhalganj, Gorakhpur, U.P., India*
[2,3]*Assistant Professor, Department of Mathematics, Noida Institute of Engineering & Technology, Greater Noida, U.P., India*

Abstract: **In the modern time, Bessel's functions appear in solving many problems of sciences and engineering together with many equations such as heat equation, wave equation, Laplace equation, Schrodinger equation, Helmholtz equation in cylindrical or spherical coordinates. In this paper, we determine Mahgoub transform of Bessel's functions. Some applications of Mahgoub transform of Bessel's functions for evaluating the integral, which contain Bessel's functions, are given.**

Keywords: **Mahgoub transform, Convolution theorem, Inverse Mahgoub transform, Bessel function.**

I. INTRODUCTION

Bessel's functions have many applications [3] to solve the problems of mathematical physics, acoustics, radio physics, atomic physics, nuclear physics, engineering and sciences such as flux distribution in a nuclear reactor, heat transfer, fluid mechanics, vibrations, hydrodynamics, stress analysis etc.

Bessel's function of order n, where $n\epsilon N$ is given by [1-5,10]

$$J_n(t) = \frac{t^n}{2^n n!}[1 - \frac{t^2}{2.(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)} + \cdots..] \ldots..(1)$$

In particular, when $n = 0$, we have Bessel's function of zero order and it is denoted by $J_0(t)$and it is given by the infinite power series

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots..(2)$$

For $n = 1$, we have Bessel's function of order one and it is denoted by $J_1(t)$and it is given by

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2.4} + \frac{t^5}{2^2.4^2.6} - \frac{t^7}{2^2.4^2.6^2.8} +..(3)$$

Equation (3) can be written as

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3.2!} + \frac{t^5}{2^5.2!.3!} - \frac{t^7}{2^7.3!.4!} + \cdots.(4)$$

For $n = 2$, we have Bessel's function of order two and it is denoted by $J_2(t)$ and it is given by

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2.4.6} + \frac{t^6}{2^2.4^2.6.8} - \frac{t^8}{2^2.4^2.6^2.8.10} + \cdots..(5)$$

The Mahgoub transform of the function $F(t)$ is defined as [6]:

$$M\{F(t)\} = v\int_0^\infty F(t)e^{-vt}dt$$
$$= H(v), t \geq 0, k_1 \leq v \leq k_2 \ldots\ldots\ldots\ldots(6)$$

where M is Mahgoub transform operator.

The Mahgoub transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Mahgoub transform of the function $F(t)$.

Mahgoub and Alshikh [7] applied Mahgoub transform for solving partial differential equations. Fadhil [8] gave the convolution for Kamal and Mahgoub transforms. Taha et. al. [9] gave the dualities between Kamal & Mahgoub integral transforms and some famous integral transforms. Aggarwal et al. [12] discussed a new application of Mahgoub transform for solving linear Volterra integral equations. Aggarwal et al. [13] solved the linear Volterra integro-differential equations of second kind using Mahgoub transform.

The object of the present study is to determine Mahgoub transform of Bessel's functions and explain the advantage of Mahgoub transform of Bessel's functions for evaluating the integral which contain Bessel's functions.

II. LINEARITY PROPERTY OF MAHGOUB TRANSFORM

If $M\{F(t)\} = H(v)$ and $M\{G(t)\} = I(v)$ then

$$M\{aF(t) + bG(t)\} = aM\{F(t)\} + bM\{G(t)\}$$

$$\Rightarrow M\{aF(t) + bG(t)\} = aH(v) + bI(v),$$

where a, b are arbitrary constants.

III. MAHGOUB TRANSFORM OF SOME ELEMENTARY FUNCTIONS [6, 8]

S.N.	$F(t)$	$M\{F(t)\} = H(v)$
1.	1	1
2.	t	$\frac{1}{v}$

3.	t^2	$\frac{2!}{v^2}$
4.	$t^n, n \in N$	$\frac{n!}{v^n}$
5.	$t^n, n > -1$	$\frac{\Gamma(n+1)}{v^n}$
6.	e^{at}	$\frac{v}{v-a}$
7.	$sinat$	$\frac{av}{v^2+a^2}$
8.	$cosat$	$\frac{v^2}{v^2+a^2}$
9.	$sinhat$	$\frac{av}{v^2-a^2}$
10.	$coshat$	$\frac{v^2}{v^2-a^2}$

IV. CHANGE OF SCALE PROPERTY OF MAHGOUB TRANSFORM

If $M\{F(t)\} = H(v)$ then

$$M\{F(at)\} = v\int_0^\infty F(at)e^{-vt}dt \ldots\ldots\ldots (7)$$

Put $at = p \Rightarrow adt = dp$ in equation (7), we have

$$M\{F(at)\} = \frac{v}{a}\int_0^\infty F(p)e^{\frac{-vp}{a}}dp = H\left(\frac{v}{a}\right)$$

Thus, if $M\{F(t)\} = H(v)$ then

$$M\{F(at)\} = H\left(\frac{v}{a}\right) \ldots\ldots\ldots\ldots\ldots. (8)$$

V. MAHGOUB TRANSFORM OF THE DERIVATIVES OF THE FUNCTION $F(t)$ [6, 8, 9]

If $M\{F(t)\} = H(v)$ then

a) $M\{F'(t)\} = vH(v) - vF(0)$
b) $M\{F''(t)\} = v^2H(v) - vF'(0) - v^2F(0)$
c) $M\{F^{(n)}(t)\} = v^nH(v) - v^nF(0) - v^{n-1}F'(0) \ldots\ldots - vF^{(n-1)}(0)$

VI. CONVOLUTION OF TWO FUNCTIONS [11]

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_0^t F(x)G(t-x)dx$$

$$= \int_0^t F(t-x)G(x)dx = G * F$$

VII. CONVOLUTION THEOREM FOR MAHGOUB TRANSFORMS [8, 12, 13]

If $M\{F(t)\} = H(v)$ and $K\{G(t)\} = I(v)$ then

$$M\{F(t) * G(t)\} = \frac{1}{v}M\{F(t)\}M\{G(t)\} = \frac{1}{v}H(v)I(v)$$

VIII. INVERSE MAHGOUB TRANSFORM [12, 13]

If $M\{F(t)\} = H(v)$ then $F(t)$ is called the inverse Mahgoub transform of $H(v)$ and mathematically it is defined as

$F(t) = M^{-1}\{H(v)\}$

where M^{-1} is the inverse Mahgoub transform operator.

IX. INVERSE MAHGOUB TRANSFORM OF SOME ELEMENTARY FUNCTIONS [12, 13]

S.N.	$H(v)$	$F(t) = M^{-1}\{H(v)\}$
1.	1	1
2.	$\frac{1}{v}$	t
3.	$\frac{1}{v^2}$	$\frac{t^2}{2!}$
4.	$\frac{1}{v^n}, n \in N$	$\frac{t^n}{n!}$
5.	$\frac{1}{v^n}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{v}{v-a}$	e^{at}
7.	$\frac{v}{v^2+a^2}$	$\frac{sinat}{a}$
8.	$\frac{v^2}{v^2+a^2}$	$cosat$
9.	$\frac{v}{v^2-a^2}$	$\frac{sinhat}{a}$
10.	$\frac{v^2}{v^2-a^2}$	$coshat$

X. RELATION BETWEEN $J_0(t)$ AND $J_1(t)$[4, 10]

$$\frac{d}{dt}J_0(t) = -J_1(t) \ldots\ldots\ldots. (9)$$

XI. RELATION BETWEEN $J_0(t)$ AND $J_2(t)$[10]

$$J_2(t) = J_0(t) + 2J_0''(t) \ldots\ldots\ldots (10)$$

XII. MAHGOUB TRANSFORM OF BESSEL'S FUNCTIONS

a) Mahgoub transform of $J_0(t)$:

Taking Mahgoub transform of equation (2), both sides, we have

$M\{J_0(t)\} = M\{1\} - \frac{1}{2^2}M\{t^2\}$
$+\frac{1}{2^2.4^2}M\{t^4\} - \frac{1}{2^2.4^2.6^2}M\{t^6\} + \cdots ..$

$$= 1 - \frac{1}{2^2}\left(\frac{2!}{v^2}\right) + \frac{1}{2^2.4^2}\left(\frac{4!}{v^4}\right) - \frac{1}{2^2.4^2.6^2}\left(\frac{6!}{v^6}\right) + \cdots \ldots ....$$

$$= \left[1 - \frac{1}{2}\left(\frac{1}{v^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{v^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{v^2}\right)^3 + \cdots \ldots \ldots \ldots .\right]$$

$$= \left(1 + \frac{1}{v^2}\right)^{-1/2} = \frac{v}{\sqrt{(1+v^2)}} \ldots \ldots ..(11)$$

b) *Mahgoub transform of* $J_1(t)$*:*

Taking Mahgoub transform of equation (9), both sides, we have

$$M\{J_1(t)\} = -M\{J_0{}'(t)\} \ldots \ldots \ldots \ldots .(12)$$

Now applying the property, Mahgoub transform of derivative of the function on equation (12), we have

$$M\{J_1(t)\} = -[vM\{J_0(t)\} - vJ_0(0)] \ldots .(13)$$

Using equation (2) and equation (11) in equation (13), we have

$$M\{J_1(t)\} = -\left[v.\frac{v}{\sqrt{(1+v^2)}} - v\right]$$

$$M\{J_1(t)\} = v - \frac{v^2}{\sqrt{(1+v^2)}} \ldots \ldots \ldots .(14)$$

c) *Mahgoub transform of* $J_2(t)$:

Taking Mahgoub transform of equation (10), both sides, we have

$$M\{J_2(t)\} = M\{J_0(t)\} + 2M\{J_0{}''(t)\} \ldots ..(15)$$

Now applying the property, Mahgoub transform of derivative of the function and using equation (11) in equation (15), we have

$$M\{J_2(t)\} = \frac{v}{\sqrt{(1+v^2)}} + 2\left[v^2M\{J_0(t)\} - v^2J_0(0) - vJ_0{}'(0)\right] \ldots \ldots .(16)$$

Using equation (2), equation (9) and equation (11) in equation (13), we have

$$M\{J_2(t)\} = \frac{v}{\sqrt{(1+v^2)}} + 2\left[v^2.\frac{v}{\sqrt{(1+v^2)}} - v^2 + vJ_1(0)\right] \ldots \ldots \ldots \ldots \ldots (17)$$

Using equation (3) in equation (17), we have

$$M\{J_2(t)\} = \frac{v}{\sqrt{(1+v^2)}} + \frac{2v^3}{\sqrt{(1+v^2)}} - 2v^2$$

$$= \frac{v + 2v^3 - 2v^2\sqrt{(1+v^2)}}{\sqrt{(1+v^2)}} \ldots \ldots ..(18)$$

d) *Mahgoub transform of* $J_0(at)$*:*

From equation (11), Mahgoub transform of $J_0(t)$ is given by

$$M\{J_0(t)\} = \frac{v}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Mahgoub transform, we have

$$M\{J_0(at)\} = \left[\frac{v/a}{\sqrt{(1+(v/a)^2)}}\right]$$

$$= \left[\frac{v}{\sqrt{(a^2+v^2)}}\right] \ldots \ldots \ldots \ldots ..(19)$$

e) *Mahgoub transform of* $J_1(at)$:

From equation (14), Mahgoub transform of $J_1(t)$ is given by

$$M\{J_1(t)\} = v - \frac{v^2}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Mahgoub transform, we have

$$M\{J_1(at)\} = \left[v/a - \frac{(v/a)^2}{\sqrt{(1+(v/a)^2)}}\right]$$

$$= \frac{v}{a}\left[1 - \frac{v}{\sqrt{(a^2+v^2)}}\right] \ldots \ldots \ldots \ldots \ldots (20)$$

f) *Mahgoub transform of* $J_2(at)$:

From equation (18), Mahgoub transform of $J_2(t)$ is given by

$$M\{J_2(t)\} = \frac{v + 2v^3 - 2v^2\sqrt{(1+v^2)}}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Mahgoub transform, we have

$$M\{J_2(at)\} = \left[\frac{v/a + 2(v/a)^3 - 2(v/a)^2\sqrt{(1+(v/a)^2)}}{\sqrt{(1+(v/a)^2)}}\right]$$

$$= \frac{v}{a^2}\left[\frac{a^2 + 2v^2 - 2v\sqrt{(a^2+v^2)}}{\sqrt{(a^2+v^2)}}\right] \ldots \ldots \ldots \ldots .(21)$$

XIII. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Mahgoub transform of Bessel's functions for evaluating the integral which contain Bessel's functions.

Application: 1 Evaluate the integral

$$I(t) = \int_0^t J_0(u) J_0(t-u)\,du \ldots\ldots\ldots.(22)$$

Applying the Mahgoub transform to both sides of equation (22), we have

$$M\{I(t)\} = M\left\{\int_0^t J_0(u) J_0(t-u)\,du\right\} \ldots.(23)$$

Using convolution theorem of Mahgoub transform on equation (23), we have

$$M\{I(t)\} = \frac{1}{v} M\{J_0(t)\} M\{J_0(t)\}$$

$$= \frac{1}{v}.\frac{v}{\sqrt{(1+v^2)}}.\frac{v}{\sqrt{(1+v^2)}} = \frac{v}{1+v^2} \ldots(24)$$

Operating inverse Mahgoub transform on both sides of equation (24), we have

$$I(t) = M^{-1}\left\{\frac{v}{1+v^2}\right\} = sint \ldots\ldots(25)$$

which is the required exact solution of equation (22).

Application: 2 Evaluate the integral

$$I(t) = \int_0^t J_0(u) J_1(t-u)\,du \ldots\ldots\ldots.(26)$$

Applying the Mahgoub transform to both sides of equation (26), we have

$$M\{I(t)\} = M\left\{\int_0^t J_0(u) J_1(t-u)\,du\right\} \ldots.(27)$$

Using convolution theorem of Mahgoub transform on equation (27), we have

$$M\{I(t)\} = \frac{1}{v} M\{J_0(t)\} M\{J_1(t)\}$$

$$= \frac{1}{v}\frac{v}{\sqrt{(1+v^2)}}.\left[v - \frac{v^2}{\sqrt{(1+v^2)}}\right]$$

$$= \frac{v}{\sqrt{(1+v^2)}} - \frac{v^2}{1+v^2} \ldots\ldots(28)$$

Operating inverse Mahgoub transform on both sides of equation (28), we have

$$I(t) = M^{-1}\left\{\frac{v}{\sqrt{(1+v^2)}}\right\} - M^{-1}\left\{\frac{v^2}{1+v^2}\right\}$$

$$= J_0(t) - cost \ldots\ldots\ldots\ldots\ldots.(29)$$

which is the required exact solution of equation (26).

Application: 3 Evaluate the integral

$$I(t) = \int_0^t J_1(t-u)\,du \ldots\ldots\ldots.(30)$$

Applying the Mahgoub transform to both sides of equation (30), we have

$$M\{I(t)\} = M\left\{\int_0^t J_1(t-u)\,du\right\} \ldots.(31)$$

Using convolution theorem of Mahgoub transform on equation (31), we have

$$M\{I(t)\} = \frac{1}{v} M\{1\} M\{J_1(t)\}$$

$$= \frac{1}{v}.1.\left[v - \frac{v^2}{\sqrt{(1+v^2)}}\right]$$

$$= 1 - \frac{v}{\sqrt{(1+v^2)}} \ldots\ldots(32)$$

Operating inverse Mahgoub transform on both sides of equation (32), we have

$$I(t) = M^{-1}\{1\} - M^{-1}\left\{\frac{v}{\sqrt{(1+v^2)}}\right\}$$

$$= 1 - J_0(t) \ldots\ldots\ldots\ldots\ldots.(33)$$

which is the required exact solution of equation (31).

XIV. CONCLUSION

In this paper, we have successfully discussed the Mahgoub transform of Bessel's functions. The given applications show that the advantage of Mahgoub transform of Bessel's functions to evaluate the integral which contain Bessel's functions.

REFERENCES

[1]. Mclachlan, N.W. Bessel functions for engineers, Longman, Oxford (1955).
[2]. Bell, W.W. Special functions for scientists and engineers, D. Van Nostrand Company LTD London.
[3]. Korenev, B.G. Bessel functions and their applications, Chapman & Hall/CRC.
[4]. Farrell, O.J. and Ross, B. Solved problems in analysis: As applied to Gamma, Beta, Legendre and Bessel function, Dover Publications Inc. Mineola, New York.
[5]. Watson, G.N. A treatise on the theory of Bessel functions, Cambridge University Press, Cambridge (1944).

[6]. Mahgoub, M.A.M. The new integral transform "Mahgoub Transform", Advances in theoretical and applied mathematics, 11(4), 391-398, (2016).
[7]. Mahgoub, M.A.M. and Alshikh,A.A. An application of new transform "Mahgoub Transform" to partial differential equations, Mathematical theory and Modeling, 7(1), 7-9, (2017) .
[8]. Fadhil, R.A. Convolution for Kamal and Mahgoub transforms, Bulletin of mathematics and statistics research, 5(4), 11-16, (2017).
[9]. Taha, N.E.H., Nuruddeen, R.I., Abdelilah, K. and Hassan, S. Dualities between "Kamal & Mahgoub Integral Transforms" and "Some Famous Integral Transforms", British Journal of Applied Science & Technology, 20(3), 1-8, (2017).
[10]. Raisinghania, M.D. Advanced differential equations, S.Chand & Company PVT LTD Ramnagar, New-Delhi.
[11]. Lokenath Debnath and Bhatta, D. Integral transforms and their applications, Second edition, Chapman & Hall/CRC (2006).
[12]. Aggarwal, S., Chauhan, R. and Sharma, N. A new application of Mahgoub transform for solving linear Volterra integral equations, Asian Resonance, 7(2), 46-48, (2018)
[13]. Aggarwal, S., Sharma, N. and Chauhan, R. Solution of linear Volterra integro-differential equations of second kind using Mahgoub transform, International Journal of Latest Technology in Engineering, Management & Applied Science, 7(5), 173-176, (2018).

AUTHORS

Sudhanshu Aggarwal received his M.Sc. degree from M.S. College, Saharanpur in 2007. He has also qualified CSIR NET examination (June-2010, June-2012, June-2013, June-2014 & June-2015) in Mathematical Sciences. He is working as an Assistant Professor in National P.G. College Barhalganj Gorakhpur. He is equipped with an extraordinary caliber and appreciable academic potency. He has around ten years of teaching experience at various engineering colleges affiliated to AKTU. His fields of interest include Integral Transform Methods, Differential and Partial Differential Equations, Integral Equations and Number Theory. He has published many research papers in national and international journals.

Nidhi Sharma received her M.Sc. degree from Banasthali University Rajasthan in 2015. She is working as an Assistant Professor in Noida Institute of Engineering & Technology, Greater Noida since 2015. She has also qualified IIT Jam exam (2013).She has published many research papers in national and international journals. Her fields of interest include Optimization, Fuzzy Logic and Topological Spaces.

Raman Chauhan has completed M.Sc. (Mathematics) from Indian Institute of Technology, Roorkee in 2011. He is working at NIET, Greater Noida since 2013 as a dynamic, successful and responsible faculty member in the department of Mathematics. He has also qualified IIT Jam exam (2009) and Gate exam (2011, 2013, 2014 & 2015).His fields of interest include Optimization, Integral Equations and Abstract Algebra.