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Finite Near-Rings, Digital Algebra and Quantum-Inspired Matrix Transformations

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ABSTRACT

Finite near-rings provide a natural algebraic framework for studying nonlinear transformations, finite-state systems, digital computation, and quantum-inspired matrix operations. This paper develops a unified framework connecting finite near-rings with digital algebra over F_2 and quantum-inspired matrix transformations. A genuine finite transformation near-ring of order 8 is constructed from zero-preserving Boolean functions, with pointwise addition and composition multiplication. Finite matrix algebras $M_2(F_2)$ and $M_2(F_3)$ are investigated as finite associative algebras containing linear transformations. The eight-element Heisenberg matrix group over F_2 is analyzed separately, including the operation $A \oplus B = A + B - I$, which produces an additive abelian group but does not automatically produce a near-ring with ordinary matrix multiplication. Pauli matrices are introduced as fundamental noncommutative operators and are used as inspiration for finite matrix transformations rather than being identified with finite fields or finite division rings. Connections with digital logic, finite automata, coding theory, graph transformations, and quantum computation are discussed. The paper emphasizes the distinction among finite groups, rings, near-rings, algebras, and division rings.

Keywords: finite near-ring; digital algebra; F_2; finite transformation; matrix algebra; Heisenberg group; Pauli matrices; quantum-inspired algebra; finite automata.

INTRODUCTION

Algebraic structures with finitely many elements are fundamental in discrete mathematics and computer science. Digital computers operate on finite sets of states, and binary computation is naturally represented by the field $F_2 = \{0,1\}$. Vectors over F_2, matrices over F_2, Boolean functions, finite automata, and finite-state transformations all provide algebraic descriptions of digital systems. A ring contains an additive abelian group together with an associative multiplication satisfying both distributive laws. A near-ring weakens these requirements and therefore permits a wider variety of noncommutative and transformation-based structures.

One particularly important source of near-rings is the set of functions from a finite group to itself, with pointwise addition and composition as multiplication. Such structures are closely connected with transformation semigroups and finite automata. A second important family comes from finite matrix algebras. Matrix multiplication is generally noncommutative, making matrices particularly suitable for modeling transformations. The Heisenberg group and Pauli matrices provide important examples of noncommutative matrix structures.

The purpose of this paper is to investigate the relationship among finite near-rings, digital transformations, finite matrix algebra, and quantum-inspired transformations. A central methodological point is that a noncommutative matrix group is not automatically a near-ring; the near-ring axioms must be verified explicitly.

Algebraic Preliminaries

A group $(G,+)$ is a set with an associative binary operation, an identity element, and inverses. If $a+b=b+a$ for all a,b, it is an abelian group.

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A ring R has an additive abelian group, associative multiplication, and both distributive laws: $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$.

In this paper we use the right near-ring convention. A right near-ring N is a set with two operations such that (N,+) is a group, multiplication is associative, and right distributivity holds:
$(a+b)c=ac+bc$.

Unlike a ring, the second distributive law need not hold.

Digital Algebra over F_2

The field $F_2=\{0,1\}$ satisfies $1+1=0$ and $-1=1$. The vector space $V=F_2^2$ contains four elements: (0,0), (1,0), (0,1), and (1,1). Vector addition is componentwise modulo 2, giving $(V,+) \cong Z_2 \times Z_2$. This is the basic state space for the finite digital transformation near-ring constructed below.

Construction of a Genuine Finite Near-Ring of Order 8

Let $V=F_2^2$ and define $N_8=\{f:V\to F_2 \mid f(0)=0\}$. There are three nonzero inputs, and each may independently be assigned 0 or 1. Hence $|N_8|=2^3=8$. Define addition pointwise by $(f+g)(x)=f(x)+g(x) \bmod 2$ and multiplication by composition, $(fg)(x)=f(g(x))$.

The Eight-Element Digital Transformation Near-Ring

Every element can be represented by the triple (f(10), f(01), f(11)). Thus the eight elements are 000,001,010,011,100,101,110,111. Pointwise addition makes N_8 an abelian group isomorphic to F_2^3. Composition is associative. Moreover, $((f+g)h)(x)=f(h(x))+g(h(x))=(fh)(x)+(gh)(x)$, so $(f+g)h=fh+gh$. Therefore N_8 is a finite right near-ring of order 8.

Addition Cayley Table

Using 0=000, a=001, b=010, c=011, d=100, e=101, f=110, and g=111, the addition table is:

+	0	a	b	c	d	e	f	g
0	0	a	b	c	d	e	f	g
a	a	0	c	b	e	d	g	f
b	b	c	0	a	f	g	d	e
c	c	b	a	0	g	f	e	d
d	d	e	f	g	0	a	b	c
e	e	d	g	f	a	0	c	b
f	f	g	d	e	b	c	0	a
g	g	f	e	d	c	b	a	0

Thus $(N_8,+) \cong F_2^3$.

Multiplication by Transformation Composition

For $f,g\in N_8$, define $(fg)(x)=f(g(x))$. Function composition is associative, so $(fg)h=f(gh)$. Composition need not be commutative, which is one of the principal reasons transformation near-rings are useful for noncommutative computation.

The Distributive Law

For $f,g,h\in N_8$, $((f+g)h)(x)=(f+g)(h(x))=f(h(x))+g(h(x))=(fh)(x)+(gh)(x)$. Hence $(f+g)h=fh+gh$. This proves right distributivity and establishes the near-ring structure under the convention adopted in this paper.

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Digital Interpretation

Each function $f:F_2^2 \rightarrow F_2$ is a Boolean function of two variables. Examples include $f(x,y)=x$, $f(x,y)=y$, $f(x,y)=x+y$, and $f(x,y)=xy$. Over F_2, addition corresponds to XOR and multiplication corresponds to AND. Finite near-rings therefore provide an algebraic framework for digital transformations.

Boolean Functions and Near-Rings

Boolean functions can be represented algebraically using polynomial expressions over F_2. The functions $x+y$ and xy correspond to XOR and AND, respectively. Composition of such transformations gives a direct connection between Boolean functions, finite transformations, and near-rings.

Matrix Algebra over F_2

The algebra $M_2(F_2)$ contains all 2×2 matrices with entries in F_2. There are $2^4=16$ elements. It is a finite associative algebra over F_2. Addition is componentwise and multiplication is ordinary matrix multiplication modulo 2. It is a ring and hence a near-ring under the convention used here, but it is not a division ring because nonzero singular matrices have no multiplicative inverses.

Digital Linear Transformations

Every $A \in M_2(F_2)$ defines $T_A:F_2^2 \rightarrow F_2^2$ by $T_A(x)=Ax$. Moreover, $T_A \circ T_B=T_{AB}$. Thus matrix multiplication represents composition of digital linear transformations.

Heisenberg Matrices over F_2

Consider $H(a,b,c)=[[1,a,c],[0,1,b],[0,0,1]]$, with $a,b,c \in F_2$. There are eight matrices. Under ordinary multiplication,

$$H(a,b,c)H(a',b',c')=H(a+a', b+b', c+c'+ab').$$

The multiplication is generally noncommutative. For $X=H(1,0,0)$ and $Y=H(0,1,0)$, $XY=H(1,1,1)$, while $YX=H(1,1,0)$. Hence $XY \neq YX$.

The Operation $A \oplus B = A + B - I$

For the Heisenberg matrices over F_2 define $A \oplus B=A+B-I$. Since $-I=I$ in characteristic 2, $A \oplus B=A+B+I$. One obtains $H(a,b,c) \oplus H(a',b',c')=H(a+a',b+b',c+c')$, so the eight matrices form an abelian group under $\oplus$. However, ordinary matrix multiplication together with this modified addition does not automatically satisfy the near-ring distributive law. Thus the structure should not be called a near-ring without an independent verification of the axioms.

Quantum-Inspired Matrix Transformations

Quantum computation uses linear operators on complex vector spaces. The Pauli matrices are $\sigma x=[[0,1],[1,0]]$, $\sigma y=[[0,-i],[i,0]]$, and $\sigma z=[[1,0],[0,-1]]$. They satisfy $\sigma x^2=\sigma y^2=\sigma z^2=I$ and have noncommutative multiplication. Their role here is inspirational: they demonstrate how matrix transformations can encode noncommuting operations.

Pauli Commutation Relations

The Pauli matrices satisfy $\sigma x \sigma y=i\sigma z$, $\sigma y \sigma z=i\sigma x$, and $\sigma z \sigma x=i\sigma y$. Reversing the order changes the sign; for example $\sigma y \sigma x=-i\sigma z$. Consequently $[\sigma x,\sigma y]=2i\sigma z \neq 0$. This is a basic example of noncommutativity in quantum operator algebra.

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Finite Analogues of Quantum Matrices

The ordinary Pauli matrices are complex matrices and therefore do not form a finite algebra. Nevertheless, their structural features can inspire investigations of finite matrix algebras such as $M_2(F_2)$ and $M_2(F_3)$. Such finite-field matrices provide algebraic analogues for studying noncommutative transformations, without claiming that they are physical quantum operators.

Finite Matrix Algebra over F_3

The algebra $M_2(F_3)$ contains $3^4=81$ matrices. It is an associative algebra over F_3. The Heisenberg family $H_3=\{H(a,b,c):a,b,c\in F_3\}$ contains $3^3=27$ matrices and forms a finite noncommutative group under matrix multiplication.

Finite Near-Rings and Finite Algebras

Finite rings are examples of near-rings under the convention used in this paper. Matrix algebras such as $M_2(F_2)$ and $M_2(F_3)$ therefore provide finite near-ring examples as well as richer associative algebra structures. Transformation near-rings are more general because they can contain nonlinear transformations and need not satisfy both distributive laws.

Near-Rings Versus Division Rings

A division ring requires every nonzero element to have a multiplicative inverse. Wedderburn's little theorem states that every finite division ring is commutative. Hence every finite division ring is a finite field. The Heisenberg matrices therefore cannot provide a finite noncommutative division ring. Their significance lies in finite noncommutative groups and matrix transformations.

Applications to Digital Logic

For $x,y\in F_2$, $x+y$ is XOR and xy is AND. Polynomial functions over F_2 therefore describe Boolean circuits. A digital transformation can be represented as $f:F_2^n\to F_2^m$, and composition represents the connection of computational stages. Near-rings provide an algebraic language for such transformations.

Applications to Finite Automata

A finite automaton has a finite state set Q. Each input symbol induces a transformation $T\_a:Q\to Q$. Products of transformations represent sequential processing. When an appropriate additive structure is introduced, transformation near-rings provide a useful algebraic model of finite-state computation.

Applications to Coding Theory

Linear codes over F_2 are vector subspaces of F_2^n. Matrices provide encoding and decoding transformations, such as $c=Gm$. Finite matrix algebras therefore form a natural framework for digital coding, while near-ring transformations can extend the framework to nonlinear operations.

Applications to Graph Theory

For a finite graph $G=(V,E)$, a transformation $f:V\to V$ can represent a graph-state operation. Composition of transformations forms a finite transformation semigroup. With suitable additive structures, near-ring methods can be applied to graph automata, network transformations, and discrete dynamical systems.

A Unified Digital-Algebraic Framework

The framework may be summarized by $F_2 \to F_2^n \to M\_n(F_2) \to$ finite transformations $\to$ finite near-rings. In parallel, Heisenberg and Pauli operators motivate the study of noncommutative matrix transformations. These themes meet in the study of finite digital and quantum-inspired algebra.

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Structural Comparison

F_2 has order 2 and is a field. F_2^2 has order 4 and is a vector space. The digital transformation near-ring N_8 has order 8. $M_2(F_2)$ has order 16 and is a finite algebra. H_2 has order 8 and is a finite noncommutative group under multiplication. H_3 has order 27. $M_2(F_3)$ has order 81. The Pauli matrices are defined over C and hence are not a finite algebra.

Proposed Research Model

A general transformation construction is $N(G)=\{f:G\to A\}$, with pointwise addition and composition multiplication whenever the chosen domains and codomains make these operations closed. For finite G and A, the resulting structure is finite. Important research questions include additive subgroups, multiplicative semigroups, ideals, zero divisors, idempotents, nilpotents, automorphisms, matrix representations, and computational applications.

Further Generalization

For $V=F\_q^n$, the number of functions $f:V\to F\_q$ satisfying $f(0)=0$ is $q\text{^}(q^n-1)$. For $q=2,n=2$ this gives $2^3=8$; for $q=2,n=3$ it gives $2^7=128$. This produces a natural hierarchy of finite digital transformation structures.

Computational Perspective

Finite near-rings can be represented by Cayley tables, transformation tables, binary matrices, truth tables, and directed graphs. Automated checking can verify associativity, identity properties, inverses, and distributive laws. This makes finite near-rings suitable for computational algebra experiments.

LIMITATIONS

Not every finite noncommutative structure is a near-ring, not every matrix group is a ring, and the operation $A\oplus B=A+B-I$ does not automatically turn the Heisenberg matrices into a near-ring. Pauli matrices over C are not finite-field objects. Quantum-inspired finite algebra should therefore be understood as an algebraic analogy rather than a direct physical model.

CONCLUSIONS

Finite near-rings provide a useful bridge between abstract algebra and finite computation. The order-8 transformation near-ring constructed from zero-preserving Boolean functions gives a genuine finite near-ring example. Matrix algebras such as $M_2(F_2)$ provide finite associative algebras, while Heisenberg matrices provide finite noncommutative matrix groups. Pauli matrices illustrate noncommuting operator algebra in quantum mechanics and inspire finite matrix analogues. The combined framework suggests further work on digital near-rings, graph-induced near-rings, finite automata, coding theory, and quantum-inspired finite algebra. A central principle is that each proposed structure must be checked against its exact axioms before it is classified as a near-ring, ring, or division ring.

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