

Q-Rung Superposition Fuzzy Subgroups Associated with Normal Subgroup

R.Nagarajan^{1*}, K.Balamurugan²

¹Department of Mathematics, J.J.College of Engineering and Technology(Sowdambikka Group of Institutions)Tiruchirappalli-620009.Tamilnadu, India.

²Department of Mathematics, M.A.M School of Engineering , Siruganur ,Tiruchirappalli-621105, Tamilnadu, India

***Corresponding author:**

DOI: <https://doi.org/10.51583/IJLTEMAS.2026.150900009>

Received: 17 September 2026; Accepted: 22 September 2026; Published: 30 September 2026

ABSTRACT

In this paper, we study the concept of q-rung fuzzy set (q-RFS) to introduce the concepts of q-rung fuzzy abelian subgroups (q-RFAS), q-rung superposition fuzzy normal subgroups (q-RSFNS) and its properties. Also, we study those concepts in terms of cartesian product q-rung superposition fuzzy sets. Finally, homomorphic images and pre image of q-rung superposition fuzzy sets are established.

Index terms: fuzzy set, q-rung superposition fuzzy set, superposition subgroup, q-rung superposition fuzzy set, Commutative group, Homomorphism, Pre-images, Support, normal subgroups.

INTRODUCTION

Uncertainty set theory (UST) is the concept and techniques that applies a form of mathematical precision to human thoughts processes that in many ways are imprecise and ambiguous by the standards of classical mathematics, uncertainty se, intuitionistic uncertainty sets, interval valued sets, bipolar uncertainty sets, and other mathematical tools are often useful approaches to deal with uncertainties. In 1965, Zadeh[21], introduced the notion of uncertainty sets.

At present, this concept has been applied to many mathematical branches. In uncertainty set theory, there are several types of uncertainty set extensions, such as intuitionistic interval valued uncertainty set, vague sets and so on. Jun et.al established an which is introduced extension of bipolar uncertainty by the Yager, launched a non-standard uncertainty set referred to as pythagorean uncertainty set which is the generalization of intuitionistic uncertainty set, the new pythagorean uncertainty sets called (3,2)-uncertainty sets introduced by [[17],[18]]. Yager and Abbasov[18] studied the pythagorean membership grades(PMGs) and consideration related to pythagorean uncertainty collections and presented the association between the PMGs and the imaginary numbers.

Reformat and Yager[11] applied the PENs in dealing with communication with respect to recommender system. Garg. H et.al[9] originated a few pythagorean uncertainty mappings and investigated their preliminary properties like derivability, continuity and differentiability in details.

In 2020, Fermatean uncertainty sets proposed by Senapati and Yager[13] can handle uncertain information for more easily in the process of decision making. In recent years, many researchers studied various properties of uncertainty set, subgroups. In 2015, Tarnauceanu[16] classified uncertainty normal subgroup of finite groups. In 2016, Onasanya[8] reviewed some anti-uncertainty properties of fuzzy subgroups. Shuaib[14] and Shahuryar[15] studied the properties of omicron uncertainty subgroup and omicron anti-uncertainty subgroup.

In 2018, Addis[1] developed uncertainty homomorphism theorems on groups. In 1986, Atanassov[2] invented intuitionistic uncertainty set. In 1996, intuitionistic uncertainty sub group was first studied by Biswas[4].

Zhan and Tan[20] introduced uncertainty M-group. Furthermore, researchers developed intuitionistic fuzzy subgroup in many ways [[14],[15]]. In 2013, Yager[17] introduced pythagorean uncertainty set, whose the sum of square of the membership degree and non-membership degree lies between 0 and 1.

In 2018, Naz et.al[17] proposed a noval approach to decision making problem using pythagorean uncertainty set. In 2019, Akram and Naz[3] applied complex pythagorean uncertainty set in decision making problems. Ejegwa[6] gave an application of pythagorean uncertainty set in career placements based on academic performance using max-min-max composition. Some results related to it were given by Peng and Yang[[9],[10]].

In this paper, we study the concept of q-rung fuzzy set (q-RFS) to introduce the concepts of q-rung fuzzy abelian subgroups (q-RFAS), q-rung superposition fuzzy normal subgroups (q-RSFNS) and its properties. Also, we study those concepts in terms of cartesian product q-rung superposition fuzzy sets. Finally, homomorphic images and pre image of q-rung superposition fuzzy sets are established.

Preliminaries and Various Basic Concepts

Definition-2.1: Fuzzy set: Let X be a non-empty set. A fuzzy set A drawn from X is defined as $A = \{(x, \mu_A(x)) / x \in X\}$, where $\mu_A: X \rightarrow [0,1]$ is the member function of the fuzzy set A .

Definition-2.2: If μ is a fuzzy subset of p , for $\alpha \in [0,1]$, then the set

$\mu_\alpha = \{x \in p / \mu(x) \geq \alpha\}$ is called level subset of p of p with respect to a fuzzy subset μ .

Definition-2.3: A fuzzy set $\mu: p \rightarrow [0,1]$ is a non-empty fuzzy subset, if μ is not constant function.

Definition-2.4: Intuitionistic fuzzy set: Let X be a non-empty set. An intuitionistic fuzzy set A in X is an object having form $A = \{(x, \mu_A(x), \gamma_A(x)) / x \in X\}$, whose the functions $\mu_A(x), \gamma_A(x): X \rightarrow [0,1]$ define respectively, the degree of membership and non-membership of the element $x \in X$ to the set A , which is a subset of X , and for every element $x \in X$, $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$. Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of $x \in A$. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFA A and $\pi_A(x) \in [0,1]$, (i.e), $\pi_A(x): X \rightarrow [0,1]$ and $0 \leq \pi_A(x) \leq 1, \forall x \in X$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IUS A or not.

Definition-2.5: Pythagorean fuzzy set : A pythagorean fuzzy set on a non-empty set X is defined as a structure $C = \{(x, \alpha_A(x), \beta_A(x)) / x \in X\}$(1)

where $\alpha_A: X \rightarrow [0,1]$ and $\beta_A: X \rightarrow [0,1]$, such that $0 \leq \alpha_A^2(x) + \beta_A^2(x) \leq 1$.

The notations $\alpha_A^2(x), \beta_A^2(x)$ are used listed of $(\alpha_A(x))^2$ and $(\beta_A(x))^2$ respectively, and the pythagorean uncertainty set (1) simply indicated by $C = (x, \alpha_A, \beta_A)$.

Definition-2.6: A group G is cyclic if it contains 'a' is a generator of G , then for every $x \in G$, there is a unique integer $n \in \mathbb{Z}$ such that $x = a^n$. Then n is called the discrete logarithm of x to base 'a' and we denoted it by $DLOGG a(x)$, $G = \{a^n / n \in \mathbb{Z}\}$. Here 'a' is called generator of G and it is denoted by $\langle a \rangle$.

Definition-2.7: q-rung superposition fuzzy subgroup: Let $A = (G, \alpha_A, \beta_A)$ be a pythagorean uncertainty set of a group G . Then A is called a q-rung superposition fuzzy subgroup (q-RSFS) of G if the following conditions hold,

$PPUS_1: \alpha_A^q(a^n * a^m) \geq \min\{\alpha_A^q(a^n), \alpha_A^q(a^m)\}$ and

$$\beta_A^q(a^n * a^m) \leq \max\{\beta_A^q(a^n), \beta_A^q(a^m)\}.$$

$$PPUS_2: \alpha_A^q(a^{-n}) \geq \alpha_A^q(a^n) \text{ and}$$

$$\beta_A^q(a^{-n}) \leq \beta_A^q(a^n), \forall n, m \in \mathbb{Z} \text{ and } a \in G.$$

Equivalently, a q-rung superposition fuzzy set $A = (X, \alpha_A, \beta_A)$ of G is a q-rung fuzzy set of G if and only if

$$\alpha_A^q(a^n a^{-m}) \geq \min\{\alpha_A^q(a^n), \alpha_A^q(a^m)\} \text{ and}$$

$$\beta_A^q(a^n a^{-m}) \leq \max\{\beta_A^q(a^n), \beta_A^q(a^m)\}, \forall n, m \in \mathbb{Z}_+.$$

Q-Rung Superposition Fuzzy Subgroup

In this section, we will discuss the concept of q-rung superposition fuzzy subgroup and its properties.

Theorem-3.1: Let $A = (X, \alpha_A, \beta_A)$ be a q-rung superposition fuzzy set of a group G .

Then we have the followings.

- (i) $N(A)$ is a subgroup of G .
- (ii) A is a q-rung superposition fuzzy normal subgroup (q-RSFNS) of G if and only if $N(A) = G$.
- (iii) A is a q-rung superposition fuzzy normal subgroup (q-RSFNS) of $N(A)$.

Proof: (i) Let $a^n, a^m \in N(A)$. Then

$$\alpha_A^q(a^{-n} x a^n) = \alpha_A^q(x) \text{ and } \beta_A^q(a^{-n} x a^n) = \beta_A^q(x), \forall x \in G \dots \dots \dots (1)$$

$$\alpha_A^q(a^{-m} y a^m) = \alpha_A^q(y) \text{ and } \beta_A^q(a^{-m} y a^m) = \beta_A^q(y), \forall y \in G \dots \dots \dots (2)$$

Put $y = a^{-n} x a^n$ in (2) and using (1) we get,

$$\alpha_A^q(a^{-m} a^{-n} x a^n a^m) = \alpha_A^q(a^{-n} x a^n) = \alpha_A^q(x) \text{ and}$$

$$\beta_A^q(a^{-m} a^{-n} x a^n a^m) = \beta_A^q(a^{-n} x a^n) = \beta_A^q(x).$$

$$\text{That is, } \alpha_A^q((a^n a^m)^{-1} x (a^n a^m)) = \alpha_A^q(x) \text{ and } \beta_A^q((a^n a^m)^{-1} x (a^n a^m)) = \beta_A^q(x)$$

$$\text{Thus, } a^n a^m = a^{n+m} \in N(A).$$

Next, change x to x^{-1} in (1), we get

$$\alpha_A^q(a^{-n} x^{-1} a^n) = \alpha_A^q(x^{-1}) = \alpha_A^q(x) \text{ and } \beta_A^q(a^{-n} x^{-1} a^n) = \beta_A^q(x^{-1}) = \beta_A^q(x).$$

That is

$$\alpha_A^q((a^n x a^{-n})^{-1}) = \alpha_A^q(a^n x a^{-n}) = \alpha_A^q(x) \text{ and}$$

$$\beta_A^q((a^n x a^{-n})^{-1}) = \beta_A^q(a^n x a^{-n}) = \beta_A^q(x).$$

That is

$$\alpha_A^q((a^{-n})^{-1} x (a^{-n})) = \alpha_A^q(a^n x a^{-n}) = \alpha_A^q(x) \text{ and } \beta_A^q((a^{-n})^{-1} x (a^{-n})) = \beta_A^q(a^n x a^{-n}) = \beta_A^q(x), x \in N(A).$$

Hence $N(A)$ is a subgroup of G .

(ii) Obvious, when $N(A) = G$, then

$$\alpha_A^q(a^{-n}xa^n) = \alpha_A^q(x) \text{ and } \beta_A^q(a^{-n}xa^n) = \beta_A^q(x), \forall x, a \in G \text{ and } n \in \mathbb{Z}.$$

Hence, A is a q -rung superposition fuzzy normal subgroup of G .

Conversely, when A is a q -rung superposition fuzzy normal subgroup of G , then

$$\alpha_A^q(a^{-n}xa^n) = \alpha_A^q(x) \text{ and } \beta_A^q(a^{-n}xa^n) = \beta_A^q(x), \forall x, a \in G \text{ and } n \in \mathbb{Z}.$$

That is, the set $\alpha_A^q(a^{-n}xa^n) = \alpha_A^q(x)$ and

the set $\beta_A^q(a^{-n}xa^n) = \beta_A^q(x), \forall x \in G = G$.

That is $N(A) = G$.

(iii) Let $a^n, a^m \in N(A)$. Then

$$\alpha_A^q(a^{-n}xa^n) = \alpha_A^q(x) \text{ and } \beta_A^q(a^{-n}xa^n) = \beta_A^q(x), \forall x \in G.$$

Put $x = a^{n+m}$, we get

$$\alpha_A^q(a^{n+m}) = \alpha_A^q(a^{-n}a^{n+m}a^n) = \alpha_A^q(a^{m+n}) = \alpha_A^q(a^m \cdot a^n) \text{ and}$$

$$\beta_A^q(a^{n+m}) = \beta_A^q(a^{-n}a^{n+m}a^n) = \beta_A^q(a^{m+n}) = \beta_A^q(a^m \cdot a^n)$$

Hence, A is a q -rung superposition fuzzy normal subgroup (q -RFNS) of $N(A)$.

Definition-3.2: Let A be a q -rung superposition fuzzy subgroup of a group G . Then A is called a q -rung superposition fuzzy abelian subgroup of G if $C_{r,s}(A)$ is an abelian subgroup of G for all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Remark-3.3: If G is an abelian group, then every q -rung superposition fuzzy subgroup (PPUS) of G is a q -rung superposition fuzzy abelian subgroup of G (q -RFAS).

Theorem-3.4: Let ' A ' be a q -rung superposition fuzzy abelian subgroup (q -RFAS) of a group G . Then the set

$$H = \{a^n \in G / \alpha_A^q(a^{n+m}) = \alpha_A^q(a^{m+n}), \beta_A^q(a^{n+m}) = \beta_A^q(a^{m+n}), \forall a^m \in G\} \text{ is an abelian subgroup of } G.$$

Proof: Since $A = (G, \alpha_A, \beta_A)$ is a q -rung superposition fuzzy abelian subgroup of a group G , $C_{r,s}(A)$ is an abelian subgroup of G for all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Clearly $H \neq \emptyset$, because $e \in H$.

Let $a^n, a^m \in H$. Then, $\alpha_A^q(a^n x) = \alpha_A^q(x a^n)$ and $\beta_A^q(a^n x) = \beta_A^q(x a^n)$

$$\alpha_A^q(a^m x) = \alpha_A^q(x a^m), \beta_A^q(a^m x) = \beta_A^q(x a^m)$$

$$\alpha_A^q((a^{n+m})x) = \alpha_A^q(a^n(a^m x))$$

$$= \alpha_A^q((a^m x)a^n)$$

$$= \alpha_A^q(a^m(x a^n))$$

$$\begin{aligned}
 &= \alpha_A^q((xa^n)a^m) \\
 &= \alpha_A^q(x(a^{n+m})) \text{ and} \\
 \beta_A^q((a^{n+m})x) &= \beta_A^q(a^n(a^m x)) \\
 &= \beta_A^q((a^m x)a^n) \\
 &= \beta_A^q(a^m(xa^n)) \\
 &= \beta_A^q((xa^n)a^m) \\
 &= \beta_A^q(x(a^{n+m})), \forall x \in G.
 \end{aligned}$$

Therefore, $a^{n+m} \in H$.

Also, let $a^n \in H$.

Now, $a^n \in H \Rightarrow \alpha_A^q(a^n x) = \alpha_A^q(x a^n)$ and $\beta_A^q(a^n x) = \beta_A^q(x a^n), \forall x \in G \dots \dots (1)$

Put $x = y^{-1}$ in (1), we get $\alpha_A^q(a^n y^{-1}) = \alpha_A^q(y^{-1} a^n)$ and $\beta_A^q(a^n y^{-1}) = \beta_A^q(y^{-1} a^n)$.

$$\begin{aligned}
 \text{Now, } \alpha_A^q(a^{-n} y) &= \alpha_A^q((a^{-n} y)^{-1}) \\
 &= \alpha_A^q((a^n y^{-1})^{-1}) \\
 &= \alpha_A^q(y a^{-n})
 \end{aligned}$$

Similarly, $\beta_A^q(a^{-n} y) = \beta_A^q(y a^{-n}), \forall y \in G$.

Thus, $a^{-n} \in H$.

So, H is a subgroup of G .

Let $a, b \in H$, without loss of generality,

Let $\alpha_A^q(a^n) = r, \beta_A^q(a^n) \leq 1 - r$ and $\alpha_A^q(a^m) = r_1, \beta_A^q(a^m) \leq 1 - r_1$.

Then, $a^n \in C_{r, r-1}(A), a^m \in C_{r_1, r_1-1}(A)$.

Let $r < r_1$. Then $\alpha_A^q(a^m) = r_1 > r$ and $\beta_A^q(a^m) \leq 1 - r_1 < 1 - r$.

$\Rightarrow a^m \in C_{r, r-1}(A)$.

Thus, $a^n, a^m \in C_{r, r-1}(A)$ and so, $a^{n+m} = a^{m+n}$.

Hence, H is an abelian subgroup of G .

Proposition-3.5: (i) If $A = (G, \alpha_A, \beta_A)$ is a q-rung superposition fuzzy abelian subgroup(q-RSFAS), then ‘ A ’ is a q-rung superposition fuzzy abelian subgroup of G .

(ii) The sets H and $C(A)$ are the same, that is $H = C(A)$.

Proof: $C(A) = \{a^n \in G / \alpha_A^q(a^n \cdot x) = \alpha_A^q(e) \text{ and } \beta_A^q(a^n \cdot x) = \beta_A^q(e), \forall x \in G\}$

$$\Rightarrow C(A) = \{a^n \in G / \alpha_A^q(a^{-n}x^{-1}a^n x) = \alpha_A^q(e) \text{ and } \beta_A^q(a^{-n}x^{-1}a^n x) = \beta_A(e), \forall x \in G\}$$

$$\Rightarrow C(A) = \{a^n \in G / \alpha_A^q((xa^n)^{-1}a^n x) = \alpha_A^q(e) \text{ and } \beta_A^q((xa^n)^{-1}a^n x) = \beta_A(e), \forall x \in G\}$$

$$\Rightarrow C(A) = H.$$

Proposition-3.6: Let $A = (G_1, \alpha_A, \beta_A)$ and $B = (G_2, \alpha_B, \beta_B)$ be a q-rung superposition fuzzy abelian subgroup(q-RSFAS) of G_1 and G_2 respectively. Then $A \times B$ is a q-rung superposition fuzzy abelian subgroup(q-RSFAS) of $G_1 \times G_2$ if and only if both A and B are q-rung superposition fuzzy abelian subgroup(q-RSFAS) of G_1 and G_2 respectively.

Proof: First, let A and B are q-rung superposition fuzzy abelian subgroups(q-RSFAS) of G_1 and G_2 respectively.

Then $C_{r,s}(A)$ and $C_{r,s}(B)$ are an abelian subgroups of G_1 and G_2 respectively for all

$$r, s \in [0,1] \text{ with } 0 \leq r + s \leq 1 \Rightarrow C_{r,s}(A) \times C_{r,s}(B) \text{ is an abelian subgroup of } G_1 \times G_2.$$

But, $C_{r,s}(A \times B) = C_{r,s}(A) \times C_{r,s}(B)$ (By definition)

Therefore, $C_{r,s}(A \times B)$ is an abelian subgroup of $G_1 \times G_2$, for all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

$\Rightarrow A \times B$ is q-rung superposition fuzzy abelian subgroup of $G_1 \times G_2$.

Conversely, let $A \times B$ is q-rung superposition fuzzy abelian subgroup of $G_1 \times G_2$.

Then, $C_{r,s}(A \times B)$ is an an abelian subgroup of $G_1 \times G_2$.

That is $C_{r,s}(A)$ and $C_{r,s}(B)$ are an abelian subgroups of G_1 and G_2 respectively.

Therefore, A and B are q-rung superposition fuzzy abelian subgroups of G_1 and G_2 respectively.

Definition-3.7: Let $A = (G, \alpha_A, \beta_A)$ be a q-rung fuzzy set(q-RFS) of a group G . Then 'A' is called q-rung superposition fuzzy subgroup of G , for all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Proposition-3.8: If G is a superposition group, then every q-rung fuzzy set of G is a

q-rung superposition fuzzy subgroup (q-RSFS) of G .

Proof: Let $G = \langle a \rangle$ be a power group and let 'A' be q-rung fuzzy set of G .

Then, $\alpha_A^q(a^n) \geq \alpha_A^q(a^{n-1}) \geq \alpha_A^q(a^{n-2}) \geq \dots \dots \dots \geq \alpha_A^q(a^2)$ and

$$\beta_A^q(a^n) \geq \beta_A^q(a^{n-1}) \geq \beta_A^q(a^{n-2}) \geq \dots \dots \dots \geq \beta_A^q(a^2), \text{ for all } n \in \mathbb{Z}.$$

Therefore, $a^m \in C_{r,s}(A)$, for some $m \in \mathbb{Z}$, then $a^m, a^{m+1}, a^{m+2}, \dots \dots \dots \in C_{r,s}(A)$.

That is $C_{r,s}(A) = \langle a^{-n} \rangle$, which is a superposition subgroup of G , for all $r, s \in [0,1]$ with

$$0 \leq r + s \leq 1.$$

Hence A is a q-rung superposition fuzzy subgroup.

Theorem-3.9: Let $f: G_1 \rightarrow G_2$ be a homomorphism of a group G_1 into a group G_2 . Let ' B ' be a q-rung superposition fuzzy abelian subgroup of G_2 . Then $f^{-1}(B)$ is a q-rung superposition fuzzy subgroup of G_1 .

Proof: Let B be a q-rung superposition fuzzy abelian subgroup of G_2 .

Therefore, $C_{r,s}(B)$ is an abelian subgroup of G_2 . For all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

It follows that, $C_{r,s}(f^{-1}(B)) = f^{-1}(C_{r,s}(B)) = \{x \in G_1 / f(x) \in C_{r,s}(B)\}$ is an abelian subgroup of G_1 .

Therefore, $f(a^n) \cdot f(a^m) = f(a^m) \cdot f(a^n)$

$$= f(a^n \cdot a^m)$$

$$= f(a^m \cdot a^n)$$

$$= f(a^{n+m})$$

$$= f(a^{m+n})$$

And so, $\alpha_B^q(f(a^n \cdot a^m)) = \alpha_B^q(f(a^m) \cdot f(a^n))$ and $\beta_B^q(f(a^n \cdot a^m)) = \beta_B^q(f(a^m) \cdot f(a^n))$

$\Rightarrow \alpha_{f^{-1}(B)}^q(a^n \cdot a^m) = \alpha_{f^{-1}(B)}^q(a^m \cdot a^n)$ and $\beta_{f^{-1}(B)}^q(a^n \cdot a^m) = \beta_{f^{-1}(B)}^q(a^m \cdot a^n)$.

Thus, $C_{r,s}(f^{-1}(B))$ is an abelian subgroup of G_1 . For all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Hence, $f^{-1}(B)$ is a q-rung superposition fuzzy abelian subgroup of G_1 .

Theorem-3.10: Let $f: G_1 \rightarrow G_2$ be surjective homomorphism of the group G_1 into a group G_2

and ' A ' a q-rung superposition fuzzy abelian subgroup of G_1 . Then $f(A)$ is a q-rung superposition fuzzy abelian subgroup of G_2 .

Proof: Since ' A ' is q-rung superposition fuzzy abelian subgroup of G_1 , for all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Let $a^p, a^q \in C_{r,s}(f(A))$. Then there exist $a^n, a^m \in G_1$ such that $f(a^n) = a^p, f(a^m) = a^q$.

Then $f(a^n), f(a^m) \in C_{r,s}(f(A))$ as $C_{r,s}(A)$ is a abelian subgroup of G_1 .

Therefore, there exists $C_{u,v}(A)$ such that $a^n, a^m \in C_{u,v}(A)$, where $u, v \in [0,1]$ with $0 \leq u + v \leq 1$. But $C_{r,s}(A)$ is a abelian subgroup.

Therefore, $a^{n+m} = a^{m+n}$

$$= f(a^n \cdot a^m)$$

$$= f(a^m \cdot a^n)$$

$$= f(a^m) \cdot f(a^n)$$

That is, $a^{p+q} = a^{q+p}$. Thus $C_{r,s}(f(A))$ is an abelian subgroup of G_2 .

Hence, $f(A)$ is a q-rung superposition fuzzy abelian subgroup of G_2 .

Theorem-3.11: Let $f: G_1 \rightarrow G_2$ be a homomorphism of a group G_1 into a group G_2 . Let ' B ' be a q-rung superposition fuzzy subgroup of G_2 . Then $f^{-1}(B)$ is a q-rung superposition fuzzy abelian subgroup of G_1 .

Proof: Since 'B' is a q-rung superposition fuzzy subgroup of G_2 , $C_{r,s}(B)$ is a superposition subgroup of G_2 . For all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Let $C_{r,s}(B) = \langle h_2 \rangle$, for some $h_2 \in G_2$. Now for $h_2 \in G_2$, there exists $h_1 \in G_1$ such that $f(h_1) = h_2$.

Thus, $C_{r,s}(B) = \langle \alpha_A(h_1) \rangle$ and so $f^{-1}(C_{r,s}(B)) = C_{r,s}(f^{-1}(B)) = h_1$.

Hence, $f^{-1}(B)$ is a q-rung superposition fuzzy subgroup of G_1 .

Theorem-3.12: Let $f: G_1 \rightarrow G_2$ be a surjective homomorphism of the group G_1 into a group G_2 and 'A' a q-rung superposition fuzzy subgroup of G_1 , then $f(A)$ is a q-rung superposition fuzzy subgroup of G_2 .

Proof: Let 'A' be a q-rung superposition fuzzy abelian subgroup of G_1 . Therefore $C_{r,s}(A)$ is a superposition subgroup of G_1 , for all $r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Let $h \in C_{r,s}(\alpha_A(A))$.

As 'f' is surjective, therefore let $h = f(h_1)$, for some $h_1 \in G_1$. We can find one $C_{r,s}(A)$ which exists for all $h_1 \in G_1$. (and hence $h \in C_{r,s}(f(A))$ such that $h \in C_{r,s}(A)$)

But $C_{r,s}(A)$ is a superposition subgroup of G_1 .

Let $C_{r,s}(A) = \langle h_1 \rangle$. So $h_1 = (h_1)^n$.

Thus $h = f(h) = f(h_1)f((h_1)^n) = (f(h_1))^n$.

That is $C_{r,s}(f(A))$ is a superposition subgroup of G_2 . Hence $f(A)$ is a q-rung superposition fuzzy subgroup of G_2 .

Support of Q-Rung Superposition Fuzzy Set

Definition-4.1: The support of a q-rung superposition fuzzy set A of X is defined to be

$$Supp_x(A) = \{a^n \in X / \alpha_A^q(a^n) > 0 \text{ and } \beta_A^q(a^n) < 1\}.$$

Clearly, $Supp_x(A)$ is $\cup\{C_{r,s}(A); \forall r, s \in [0,1] \text{ with } r + s \leq 1\}$.

Proposition-4.2: For $f: X \rightarrow Y$ and q-rung superposition fuzzy set A, B of X and Y respectively, we have

1. $f(Supp_x(A)) \subseteq Supp_x(A)$, equality holds if 'f' is bijective.
2. $f^{-1}(Supp_x(B)) = Supp_x(f^{-1}(B))$.

Proposition-4.3: If 'A' is a non-zero q-rung superposition fuzzy set of a group, the $Supp_G(A)$ is a subgroup of G. The following example shows that the converse of the proposition is not true.

Example-4.4: Let $G = (R, +)$ be a group of real numbers under addition.

Define the q-rung superposition fuzzy set A on G by

$$\alpha_A(a^n) = \begin{cases} 0.32, & \text{if } a^n = 1 \\ 0.79, & \text{if } a^n \in Q - \{1\} \\ 0, & \text{if } a^n \in R - Q \end{cases}$$

$$\beta_A(a^n) = \begin{cases} 0.32, & \text{if } a^n = 1 \\ 0.29, & \text{if } a^n \in Q - \{1\} \\ 0, & \text{if } a^n \in R - Q \end{cases}$$

Clearly, A is not a q -rung superposition fuzzy set of G , but $\text{Supp}_G(A) = Q$ is a subgroup of G .

Proposition-4.5: Let $A = (G, \alpha_A, \beta_A)$ be a q -rung superposition fuzzy set of G . Then, we have the followings.

- (i) If $A = (G, \alpha_A, \beta_A)$ be a q -rung superposition fuzzy set of G and N is a subgroup of G , then A/N is also a q -rung superposition fuzzy set of N .
- (ii) If A/N is the restriction of the q -rung superposition fuzzy set A of a group on the subgroup N of G , then $\text{Supp}_N(A/N) = \text{Supp}_G(A) \cap N$.
- (iii) If ' A ' is a q -rung superposition fuzzy set of G and N is a subgroup of G , then A/N is also a q -rung superposition fuzzy set of N if and only if N is a subgroup of G .

Proposition-4.6: Let $f: G_1 \rightarrow G_2$ be a homomorphism of a group G_1 into a group G_2 . Then, we have the followings:

- (i) If ' A ' is a q -rung superposition fuzzy abelian subgroup (q-RSFAS) of G_1 , then $f(A)$ is a q -rung superposition fuzzy abelian subgroup of G_2 .
- (ii) If ' A ' is a q -rung superposition fuzzy subgroup (q-RSFS) of G_1 , then $f(A)$ is a q -rung superposition fuzzy subgroup of G_2 .
- (iii) If ' A ' is a q -rung superposition fuzzy abelian subgroup (q-RSFAS) on G_2 , then $f^{-1}(A)$ is a q -rung superposition fuzzy abelian subgroup of G_1 .
- (iv) If ' A ' is a q -rung superposition fuzzy subgroup (q-RSFS) on G_2 , then $f^{-1}(A)$ is a q -rung superposition fuzzy subgroup of G_1 .

CONCLUSION

In this work, we have introduced the concept q -rung superposition fuzzy set, equivalent sets and established these in properties. We also given certain counter examples to prove some properties of them.

As interesting kinds, we have introduced and studied the concepts q -rung superposition fuzzy abelian subgroups and q -rung superposition fuzzy normal subgroups. Finally, homomorphic images and pre-images of q -rung superposition fuzzy sets are established.

REFERENCES

- Gezahagne Mulat Addis, "Fuzzy Homomorphism Theorems on Groups," Korean Journal of Mathematics, vol. 26, no. 3, pp. 373-385, 2018. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- Krassimir T. Atanassov, "Intuitionistic Fuzzy Sets," Fuzzy Sets and Systems, vol. 20, no. 1, pp. 87-96, 1986. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- Muhammad Akram, and Sumera Naz, "A Novel Decision-Making Approach under Complex Pythagorean Fuzz Environment," Mathematical and Computational Applications, vol. 24, no. 3, pp. 1-33, 2019. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- Ranjit Biswas, "Intuitionistic Fuzzy Subgroup," Mathematical Forum, vol. 10, pp. 39-44 1989. [[Google Scholar](#)] [[Publisher Link](#)]
- R. Biswas, "Fuzzy Subgroups and Anti-Fuzzy Subgroups," Fuzzy Sets and Systems, vol. 35, no. 1, pp. 121-124, 1990. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- Paul Augustine Ejegwa, "Pythagorean Fuzzy Set and Its Applications in Career Placements Based on Academics Performance Using Max-Min-Max Composition," Complex & Intelligent Systems, vol. 5, pp. 165-175, 2019. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]

7. Sumera Naz, Samina Ashraf, and Muhammad Akram, "A Novel Approach to Decision Making with Pythagorean Fuzzyinformation," Mathematics, vol. 6, no. 6, pp. 1-33, 2018. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
8. B.O. Onasanya, "Review of Some Anti-Fuzzy Properties of Some Fuzzy Subgroups," Annals of Fuzzy Mathematics and Informatics, vol. 11, no. 6, pp. 899-904, 2016. [[Google Scholar](#)] [[Publisher Link](#)]
9. Xindong Peng, and Yong Yang, "Some results for Pythagorean fuzzy Sets," International Journal of Intelligent Systems, vol. 30, pp. 1133-1160, 2015. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
10. Xindong Peng, and Yong Yang, "Fundamental Properties of Interval-Valued Pythagorean Fuzzy Aggregation Operators," International Journal of Intelligent Systems, vol. 31, pp. 444-487, 2016. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
11. Marek Z. Reformat, and Ronald R. Yager, "Suggesting Recommendations Using Pythagorean Fuzzy Sets Illustrated Using Netflix Movie Data," Information Processing and Management of Uncertainty in Knowledge-Based Systems, pp. 546-556, 2014. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
12. Tapan Senapati, and Ronald R. Yager, "Fermatean Fuzzy Sets," Journal of Ambient Intelligence and Humanized Computing, vol. 11, no. 2, pp. 663-674, 2020. [[CrossRef](#)][[Google Scholar](#)] [[Publisher Link](#)]
13. Tapan Senapati, and Ronald R. Yager, "Fermatean Fuzzy Weighted veraging/Geometric Operators and Its Application in Multi Criteria Decision-Making Methods," Engineering Applications of Artificial Intelligence, vol. 85, pp. 112-121, 2019.[[CrossRef](#)][[Google Scholar](#)] [[Publisher Link](#)]
14. Umer Shuaib, Muhammad Shaheryar, and Waseem Asghar, "On Some Characterizations of O-Fuzzy Subgroups," International Journal of Mathematics and Computer Science, vol. 13, no.2, pp. 119-131, 2018. [[Google Scholar](#)] [[Publisher Link](#)]
15. Umer Shuaib, and Muhammad Shaheryar, "On Some Properties of O-Anti Fuzzy Subgroups," International Journal of Mathematics and Computer Science, vol. 14, no. 1, pp. 215-230, 2019. [[Google Scholar](#)] [[Publisher Link](#)]
16. Marius Tarnauceanu, "Classifying Fuzzy Normal Subgroups \ of Finite Groups," Iranian Journal of Fuzzy Systems, vol. 12, no. 2, pp. 107-115, 2019. [[CrossRef](#)][[Google Scholar](#)] [[Publisher Link](#)]
17. Ronald R. Yager, "Properties and Application of Pythagorean Fuzzy Sets," Imprecision and Uncertainty in Information Representation and Processing, vol. 332, pp. 119-136,2016. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
18. Ronald R. Yager, and Ali M. Abbasov, "Pythagorean Membership Grades, Complex Numbers and Decision Making," International Journal of Intelligent System, vol. 28, pp. 436-452, 2013. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
19. Ronald R. Yager, "Pythagorean Fuzzy Subsets," 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, pp. 57-61, 2013. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
20. J. Zhan, and Z. Tan, Intuitionistic M-Fuzzy Groups, Soochow Journal of Mathematics, vol. 30, pp. 85-90, 2004.[[Google Scholar](#)]
- L.A. Zadeh, "Fuzzy Sets," Information and Control, vol. 8, no. 3, pp. 338-353, 1965. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]