

The Minimum Covering Schultz Energy

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DOI: <https://doi.org/10.51583/IJLTEMAS.2026.150900031>

Received: 16 September 2026; Accepted: 21 September 2026; Published: 03 October 2026

ABSTRACT

In this manuscript, we introduce the concept of minimum covering schultz energy of a graph using adjacency, vertex degree and distance between the vertices. This measure extends the classical schultz energy by incorporating the idea of a minimum vertex covering providing a more informative representation of graph structure. We compute the minimum covering schultz energy for several standard graphs including complete graph, cocktail party graph, star graph and complete bipartite graph. Furthermore, we establish general lower and upper bounds for the minimum covering schultz energy. These results contribute to a better understanding of the relationship between graph coverings, distance-based measures and spectral properties.

Keywords: Schultz Energy, Schultz Matrix, Schultz Eigenvalues.

INTRODUCTION

During the year 1978 I. Gutman [6] introduced the concept of graph energy while studying the energies of carbon atoms in conjugated hydrocarbons. In this work we consider only simple graphs that is graphs without multiple edges or loops. Let $\rho_1, \rho_2, \rho_3, \dots, \rho_n$ represents eigenvalues of a graph G , which are arranged in descending order and the adjacency matrix is $A = (a_{ij})$. Sum of eigenvalues is equal to zero, since A is real and symmetric. $\varepsilon(G)$ of G is stated as sum of absolute eigenvalues of G .

$$\varepsilon(G) = \sum_{i=1}^n |\rho_i|$$

In many reviews [3, 4, 7] and citations in graph energy we can observe the theories on the mathematical concepts. In these papers [9, 13], we can find many lower and upper bounds for energy of a graph. In molecular orbital theory of conjugated molecules [8, 5] there are chemical applications based on the observations of graph energy. The schultz index is a new topological index introduced by H. P. Schultz [10] in the year 1989 and is stated as

$$\mathfrak{S}(G) = \sum_{\{\dot{v}_i, \dot{v}_j\} \subseteq V(G)} [d(\dot{v}_i) + d(\dot{v}_j)]d(\dot{v}_i, \dot{v}_j)$$

where $d(\dot{v}_i, \dot{v}_j)$ is the shortest path between two vertices $\dot{v}_i$ and $\dot{v}_j$, $d(\dot{v}_i)$ is the vertex of degree $d(\dot{v}_i)$ in G , similarly $d(\dot{v}_j)$. Motivated by this Rajesh Kanna et.al [14] defined schultz energy of graphs and now we are defining minimum covering schultz energy.

RESULTS RELATED TO BOUNDS ON ENERGY OF A GRAPH

Theorem 1. Let a_i and b_i where $1 \leq i \leq n$ are non-negative real numbers. Then

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_i b_i\right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2$$

where $M_1 = \max_{1 \leq i \leq n} (a_i)$, $M_2 = \max_{1 \leq i \leq n} (b_i)$, $m_1 = \min_{1 \leq i \leq n} (a_i)$ and $m_2 = \min_{1 \leq i \leq n} (b_i)$

Theorem 2. Let a_i and b_i where $1 \leq i \leq n$ be positive real numbers then

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left(\sum_{i=1}^n a_i b_i\right)^2$$

where M_i and m_i are similar to previous theorem.

Theorem 3. Let a_i and b_i where $1 \leq i \leq n$ be positive real numbers then

$$\left| n \sum_{i=1}^n a_i b_i - \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) \right| \leq \alpha(n)(A - a)(B - b)$$

where a, b, A and B are real constants and $1 \leq i \leq n$ for all i . Also, $a \leq a_i \leq A$, $b \leq b_i \leq B$ and

$$\alpha(n) = n \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$$

Theorem 4. Let $a_1, a_2, \dots, a_n$ are non-negative numbers then

$$n \left(\frac{1}{n} \sum_{i=1}^n a_i - \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}} \right) \leq n \sum_{i=1}^n a_i - \left(\sum_{i=1}^n \sqrt{a_i} \right)^2 \leq n(n-1) \left(\frac{1}{n} \sum_{i=1}^n a_i - \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}} \right)$$

Theorem 5. Let a_i and b_i where $1 \leq i \leq n$ be positive real numbers then

$$\sum_{i=1}^n b_i^2 + rR \sum_{i=1}^n a_i^2 \leq (r+R) \left(\sum_{i=1}^n a_i b_i \right)$$

where r and R are real constants for all i and $1 \leq i \leq n$ holds for $ra_i \leq b_i \leq Ra_i$

THE MINIMUM COVERING ENERGY

Let $G = (V, E)$ be a simple graph and let $C \subseteq V$ be a minimum vertex cover of G . The minimum covering matrix of G denoted by $A_C(G) = (a_{ij})$ is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1, & \text{if } i = j \text{ and } v_i \in C \\ 0, & \text{otherwise} \end{cases}$$

The adjacency matrix of G with diagonal entries equal to 1 for vertices in the minimum vertex cover. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $A_C(G)$ then the minimum covering energy of the graph G is defines as

$$\varepsilon_C(G) = \sum_{i=1}^n |\lambda_i|$$

The minimum covering energy is the sum of absolute eigenvalues of a modified adjacency matrix where vertices in a minimum vertex cover are given an extra weight.

SCHULTZ ENERGY OF A GRAPH

Let $G = (V, E)$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. The Schultz matrix of G denoted by $S(G)$ is defined as:

$$s_{ij} = (d_i + d_j) d(i, j)$$

where d_i and d_j are the degrees of vertices v_i and v_j and $d(i, j)$ is the distance between v_i and v_j . The schultz energy of the graph G is defined as the sum of the absolute values of the eigenvalues of the schultz matrix and is as follows:

$$\varepsilon_S(G) = \sum_{i=1}^n |\lambda_i|$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the schultz matrix $S(G)$. The schultz energy measures a graphs structural property by combining vertex degrees and distances which analyzes the spectral behavior of the resulting adjacency matrix.

THE MINIMUM COVERING SCHULTZ ENERGY OF A GRAPH

Let $G = (V, E)$ be a graph without loops and multiple edges and C is said to be a covering set of G , if all the lines or edges of G is incident to one or more vertex of C where $C \subset V$. The minimum cardinality of any covering set is known as minimum covering set. Let us denote a minimum covering set of a graph G by C . For the graph of order $n \times n$ the minimum covering schultz matrix is defined by $A_S^C(G) = (s_{ij})_{n \times n}$ where

$$s_{ij} = \begin{cases} [d(v_i) + d(v_j)]d(v_i, v_j), & \text{if } v_i v_j \in E(G) \\ 1, & \text{if } i = j \text{ and } v_i \in C \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $S^C(G)$ is denoted by $f_n(G, \lambda) = \det(\lambda I - A_S^C(G))$. The eigenvalues of $A_S^C(G)$ is the minimum covering schultz eigenvalues of the graph G . In the schultz matrix $A_S^C(G)$, eigenvalues are real numbers. Since it is real and symmetric we label them in descending order $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The minimum covering schultz energy of G is defined as

$$\varepsilon_S^C(G) = \sum_{i=1}^n |\lambda_i|$$

The minimum covering schuhltz energy depends on the minimum covering set of the graph.

EXAMPLES RELATED TO $\varepsilon_S^C(G)$

Bull Graph: Let G be a simple undirected bull graph with $n = 5$ vertices and $m = 5$ edges. The graph is in the form of a triangle with two disjoint pendant edges which resembles a bull's head with horns as shown in the Figure 1.1

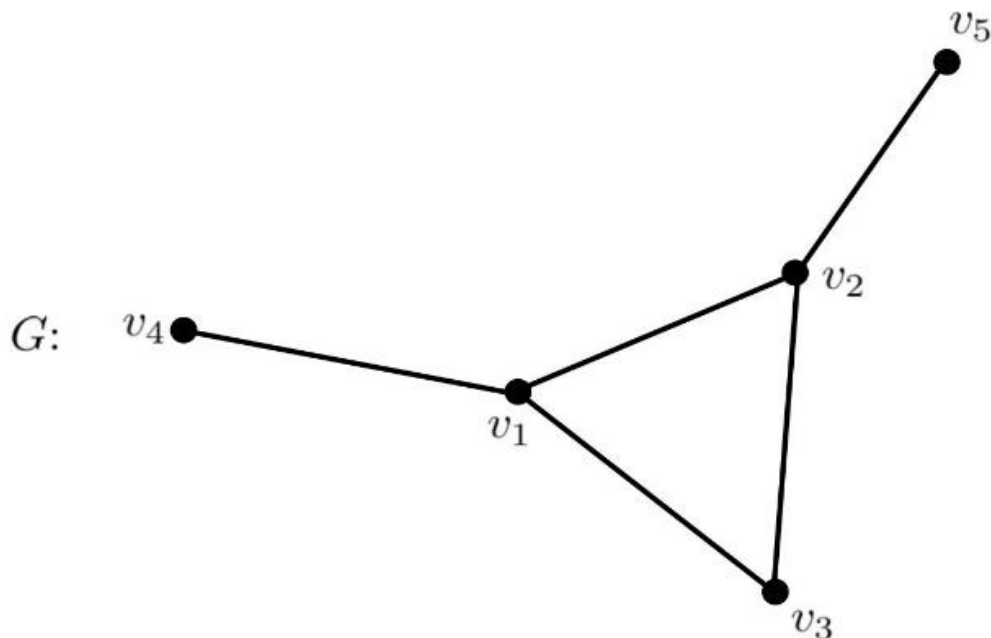


Figure 1.1. Bull Graph with 5 vertices and 5 edges

The vertex set $\{v_1, v_2\}$ is each of degree 3, $\{v_3\}$ is each of degree 2 and the vertex set $\{v_4, v_5\}$ is each of degree 1. The minimum vertex cover size of the bull graph is 2. Let $C = \{v_1, v_2\}$ be the minimum covering set and the associated covering Schultz matrix $A_S^C(G)$ is as follows:

$$A_S^C(G) = \begin{pmatrix} 1 & 6 & 5 & 4 & 8 \\ 6 & 1 & 5 & 8 & 4 \\ 5 & 5 & 0 & 6 & 6 \\ 4 & 8 & 6 & 0 & 6 \\ 8 & 4 & 6 & 6 & 0 \end{pmatrix}_{5 \times 5}$$

The characteristic polynomial of the schultz matrix $A_S^C(G)$ is $(\lambda^2 + 11\lambda + 14)(\lambda^3 - 13\lambda^2 - 224\lambda - 636) = 0$. The minimum covering schultz eigenvalues are $\lambda_1 = 23.6223, \lambda_2 = -9.5311, \lambda_3 = -6.4446, \lambda_4 = -4.1777$, and $\lambda_5 = -1.4689$. Therefore, the minimum covering Schultz energy of bull graph is $\varepsilon_S^C(G) \approx 45.2446$.

Triangular oxide network $TOX(n)$: The minimum covering schultz energy of a graph of triangular oxide network $TOX(n)$ when $n = 2$ approximately 83.0401.

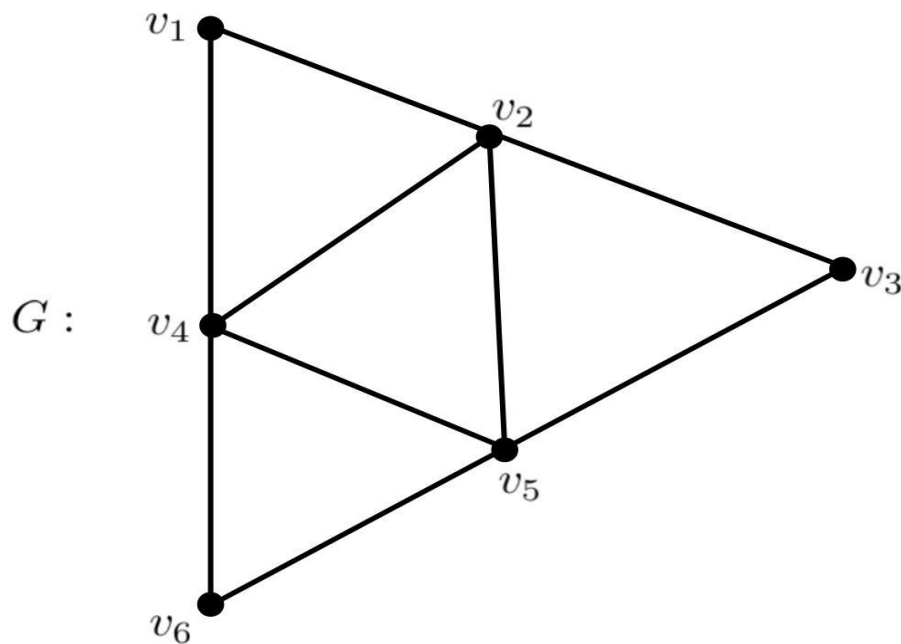


Figure 1.2. Triangular Oxide Network $TOX(2)$

A Triangular oxide network $TOX(n)$ is a graph model of a silicate structure where the silicon ions are removed, leaving only the triangularly arranged oxygen atoms. For a given length n , it consists of $\frac{n^2+3n+2}{2}$ vertices and $\frac{3(n^2+n)}{2}$ edges (Oxygen atoms). The $TOX(n)$ network is a mathematical structure to study the properties of certain oxides by analyzing its topology. It is a topic of study in chemical graph theory and is used to analyze the properties of specific oxide compounds by using mathematical tools called topological indices.

The vertex set $\{v_1, v_3, v_6\}$ is each of degree 2 and $\{v_2, v_4, v_5\}$ is each of degree 4. The minimum vertex cover size of Triangular oxide network $TOX(2)$ is 3. Let $C = \{v_2, v_4, v_5\}$ be the minimum covering set and the associated covering schultz matrix $A_S^C(G)$ is as follows:

$$A_S^C(G) = \begin{pmatrix} 0 & 6 & 8 & 6 & 12 & 8 \\ 6 & 1 & 6 & 8 & 8 & 16 \\ 8 & 6 & 0 & 12 & 6 & 8 \\ 6 & 8 & 12 & 1 & 8 & 6 \\ 12 & 8 & 6 & 8 & 1 & 6 \\ 8 & 16 & 8 & 6 & 6 & 0 \end{pmatrix}_{6 \times 6}$$

The characteristic polynomial of the schultz matrix $A_S^C(G)$ is $(\lambda^2 + 15\lambda + 20)(\lambda^4 - 18\lambda^3 - 891\lambda^2 - 4852\lambda + 6400) = 0$. The minimum covering schultz eigenvalues are $\lambda_1 = 41.9255, \lambda_2 = -16.6365, \lambda_3 = -13.5208, \lambda_4 = -8.3836, \lambda_5 = -1.4792$ and $\lambda_6 = 1.0942$. Therefore, the minimum covering Schultz energy of triangular oxide network $TOX(2)$ is $\varepsilon_S^C(G) \approx 83.0401$.

Petersen graph: Let G be a Petersen graph, which is a simple, undirected graph with 10 vertices and 15 edges, as represented in Figure 1.3. The vertex set $\{v_1, v_2, \dots, v_{10}\}$ is each of degree 3.

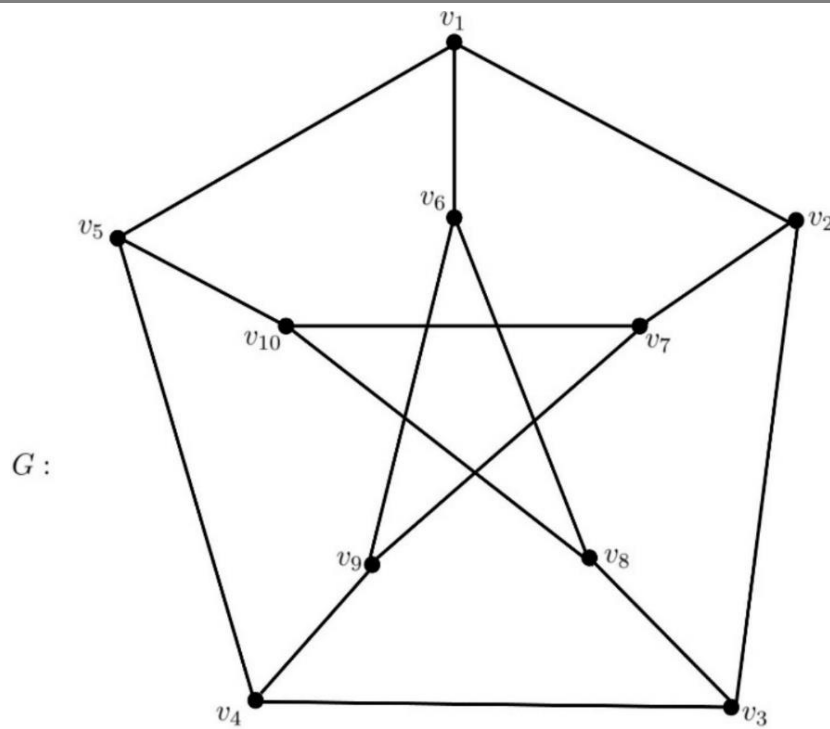


Figure 1.3. Petersen Graph with 10 vertices and 15 edges

The minimum vertex cover size of Petersen graph is 6. Let $C = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the minimum covering set and the associated covering schultz matrix $A_S^C(G)$ is as follows:

$$A_S^C(G) = \begin{pmatrix} 1 & 6 & 12 & 12 & 6 & 6 & 12 & 12 & 12 & 12 \\ 6 & 1 & 6 & 12 & 12 & 12 & 6 & 12 & 12 & 12 \\ 12 & 6 & 1 & 6 & 12 & 12 & 12 & 6 & 12 & 12 \\ 12 & 12 & 6 & 1 & 6 & 12 & 12 & 12 & 6 & 12 \\ 6 & 12 & 12 & 6 & 1 & 12 & 12 & 12 & 12 & 6 \\ 6 & 12 & 12 & 12 & 12 & 1 & 12 & 6 & 6 & 12 \\ 12 & 6 & 12 & 12 & 12 & 12 & 0 & 12 & 6 & 6 \\ 12 & 12 & 6 & 12 & 12 & 6 & 12 & 0 & 12 & 6 \\ 12 & 12 & 12 & 6 & 12 & 6 & 6 & 12 & 0 & 12 \\ 12 & 12 & 12 & 12 & 6 & 12 & 6 & 6 & 12 & 0 \end{pmatrix}_{10 \times 10}$$

The characteristic polynomial of the schultz matrix is $A_S^C(G) (\lambda + 17)(\lambda^4 + 34\lambda^3 + 271\lambda^2 - 306\lambda + 36)(\lambda^5 - 57\lambda^4 - 2787\lambda^3 - 23743\lambda^2 - 37998\lambda - 12060) = 0$. The minimum covering schultz eigenvalues are $\lambda_1 = 206.1401, \lambda_2 = -165.1401, \lambda_3 = -17.8760, \lambda_4 = -17.1338, \lambda_5 = -17, \lambda_6 = 15, \lambda_7 = 2, \lambda_8 = -1, \lambda_9 = 0.8760$ and $\lambda_{10} = 0.1338$. Therefore, the minimum covering Schultz energy of Petersen graph is $\varepsilon_S^C(G) \approx 442.2998$.

Fundamental Properties on Minimum Covering Schultz Eigenvalues of $A_S^C(G)$

Theorem 1. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G)$ be the vertex set and edge set of a graph G and C be the minimum covering set. For minimum covering schultz matrix $A_S^C(G)$, if $\lambda_1, \lambda_2, \dots, \lambda_n$ represents eigenvalues then

$$(i) \sum_{i=1}^n \lambda_i = |C|$$

$$(ii) \sum_{i \geq 1}^n \lambda_i^2 = |C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2$$

Proof: (i) We know that the trace of $A_S^C(G)$ is the sum of the eigenvalues of $A_S^C(G)$

$$\sum_{i \geq 1}^n \lambda_i = \sum_{i \geq 1}^n s_{ii} = |C|$$

where $|C|$ is the minimum covering set.

(ii) The trace of $(A_S^C(G))^2$ is sum of squares of the eigenvalues of $A_S^C(G)$. Then

$$\begin{aligned} \sum_{i \geq 1}^n \lambda_i^2 &= \sum_{j \geq 1}^n \sum_{i \geq 1}^n s_{ij} s_{ji} = \sum_{i \geq 1}^n (s_{ii})^2 + \sum_{j \neq i} s_{ij} s_{ji} \\ &= \sum_{i \geq 1}^n (s_{ii})^2 + 2 \sum_{j > i} (s_{ij})^2 \\ &= \sum_{i \geq 1}^n (s_{ii})^2 + 2 \sum_{j > i} ([d(v_i) + d(v_j)] d(v_i, v_j))^2 \\ &= |C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 \\ &= 2M, \text{ where } M = \frac{|C|}{2} + \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 \end{aligned}$$

BOUNDS ON $\epsilon_S^C(G)$

For ordinary energy of a graph McLelland's [13] gave both upper and lower bounds. Similar bounds for $\epsilon_S^C(G)$ are given in the following theorems.

Theorem 1. Let $G = (n, m)$ be a simple graph and $P = \det(A_S^C(G))$ then

$$\sqrt{|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 + n(n-1)P^{\frac{2}{n}}} \leq \epsilon_S^C(G) \leq \sqrt{n \left(|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 \right)}$$

Proof: The Cauchy-Schwarz inequality is

$$\left(\sum_{i \geq 1}^n a_i b_i \right)^2 \leq \left(\sum_{i \geq 1}^n a_i^2 \right) \cdot \left(\sum_{i \geq 1}^n b_i^2 \right)$$

If $a_i = 1$ and $b_i = |\lambda_i|$ in the above inequality

$$\begin{aligned} \left(\sum_{i \geq 1}^n |\lambda_i| \right)^2 &\leq \left(\sum_{i \geq 1}^n 1 \right) \left(\sum_{i \geq 1}^n |\lambda_i|^2 \right) \\ (\varepsilon_S^C(G))^2 &\leq n \left(|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2 \right) \\ \Rightarrow \varepsilon_S^C(G) &\leq \sqrt{n \left(|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2 \right)} \end{aligned}$$

Since geometric mean is less than or equal to arithmetic mean, we have

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{j \neq i} |\lambda_i| |\lambda_j| &\geq \left(\prod_{j \neq i} |\lambda_i| |\lambda_j| \right)^{\frac{1}{n(n-1)}} \\ \frac{1}{n(n-1)} \sum_{j \neq i} |\lambda_i| |\lambda_j| &= \left(\prod_{i \geq 1} |\lambda_i|^{2(-1+n)} \right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{i \geq 1} |\lambda_i| \right)^{\frac{2}{n}} = \left(\prod_{i \geq 1} \lambda_i \right)^{\frac{2}{n}} \\ &= \left| \det(A_S^C(G)) \right|^{\frac{2}{n}} = P_n^{\frac{2}{n}} \\ \sum_{j \neq i} |\lambda_i| |\lambda_j| &\geq n(n-1) P_n^{\frac{2}{n}} \end{aligned}$$

$$\begin{aligned} (\varepsilon_S^C(G))^2 &= \left(\sum_{i \geq 1}^n |\lambda_i| \right)^2 = \sum_{j \neq i} |\lambda_i| |\lambda_j| + \sum_{i \geq 1}^n |\lambda_i|^2 \\ (\varepsilon_S^C(G))^2 &\geq 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2 + (n^2 - n) P_n^{\frac{2}{n}} + |C| \\ \Rightarrow \varepsilon_S^C(G) &\geq \sqrt{2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2 + (n^2 - n) P_n^{\frac{2}{n}} + |C|} \end{aligned}$$

From the above result we conclude that

$$\sqrt{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2 + n(n-1) P_n^{\frac{2}{n}}} \leq \varepsilon_S^C(G) \leq \sqrt{n \left(|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2 \right)}$$

Theorem 2. If $\lambda_1(G)$ denote the largest minimum covering Schultz eigenvalue of $A_S^C(G)$, then

$$\lambda_1(G) \geq \frac{|C| + 2 \sum_{i < j} (d_i + d_j) d_{ij}}{n}$$

Proof: For $X \neq 0$, we have by [6]

$$\lambda_1(A) = \max_{X \neq 0} \left(\frac{XAX^T}{XX^T} \right)$$

$$\therefore \lambda_1(G) \geq \frac{JAJ^T}{JJ^T} = \frac{|C| + 2 \sum_{j > i} (d_i + d_j) d_{ij}}{n} \text{ here } J \text{ represents unit column matrix.}$$

An upper bound for $\varepsilon_S^C(G)$ is given in the following theorem. Upper bound for energy of a graph by Koolen and Moulton's [11].

Theorem 3. If $G = (n, m)$ is a simple graph with $\left(\frac{|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2}{n} \right) \geq 1$ then

$$\varepsilon_S^C(G) \leq \left(\frac{|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2}{n} \right) + \sqrt{(n-1) \left[|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 - \left(\frac{|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2}{n} \right)^2 \right]}$$

Proof: The Cauchy-Schwarz inequality is

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \cdot \left(\sum_{i=1}^n b_i^2 \right)$$

Put $a_i = 1$ and $b_i = |\lambda_i|$ in the above inequality

$$\left(\sum_{i=1}^n |\lambda_i| \right)^2 \leq \left(\sum_{i=1}^n |\lambda_i|^2 \right) \left(\sum_{i=1}^n 1 \right)$$

$$(\varepsilon_S^C(G) - \lambda_1)^2 \leq (n-1) \left(|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 - \lambda_1^2 \right)$$

$$\Rightarrow \varepsilon_S^C(G) \leq \lambda_1 + \sqrt{(n-1) \left(|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 - \lambda_1^2 \right)}$$

Let $f(x) = x + \sqrt{(n-1) \left(|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 - x^2 \right)}$. For decreasing function $f'(x) \leq 0$ we have,

$$1 - \frac{x(n-1)}{\sqrt{(n-1) \left(2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 - x^2 \right)}} \leq 0$$

$$\Rightarrow x \geq \sqrt{\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n}}$$

Since $\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n} \geq 1$ we have

$$\sqrt{\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n}} \leq \frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n} \leq \lambda_1$$

$$f(\lambda_1) \leq f\left(\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n}\right)$$

$$\varepsilon_S^C(G) \leq f(\lambda_1) \leq f\left(\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n}\right)$$

$$\Rightarrow \varepsilon_S^C(G) \leq f\left(\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n}\right)$$

Thus, from the above we conclude that

$$\varepsilon_S^C(G) \leq \left(\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n}\right) + \sqrt{(n-1) \left[|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2 - \left(\frac{|C| + 2 \sum_{j>i} (d_i + d_j)^2 d_{ij}^2}{n}\right)^2\right]}$$

In the following theorem, Milovanovic [12] bounds for minimum covering schultz energy of a graph given.

Theorem 4. Let $G = (n, m)$ be a simple graph where n and m are the number of vertices and edges. Let $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ be the minimum covering schultz eigenvalues of $A_S^C(G)$ arranged in the descending order then

$$\varepsilon_S^C(G) \geq \sqrt{n \left(|C| + 2 \sum_{j<i} (d_i + d_j)^2 d_{ij}^2 \right) - \alpha(n)(-|\lambda_n| + |\lambda_1|)^2}$$

Here $\alpha(n) = n \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$ and $[n]$ indicates the integral part of a real number.

Proof: Let $A \geq a_i \geq a$ and $B \geq b_i \geq b \forall i = 1, 2, 3, \dots, n$ where $a, a_1, a_2, \dots, a_n, A$ and $b, b_1, b_2, \dots, b_n, B$ are the real numbers.

$$\alpha(n)(-a + A)(-b + B) \geq \left| n \sum_{i \geq 1}^n a_i b_i - \left(\sum_{i \geq 1}^n a_i \right) \left(\sum_{i \geq 1}^n b_i \right) \right|$$

where $\alpha(n) = n \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$ and equality holds if and only if $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}_3 = \dots = \mathbf{b}_n$ and $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}_3 = \dots = \mathbf{a}_n$. If $\mathbf{b}_i = |\lambda_i|$, $\mathbf{a}_i = |\lambda_i|$, $\mathbf{B} = \mathbf{A} = |\lambda_1|$ and $\mathbf{b} = \mathbf{a} = |\lambda_n|$, then

$$\left| n \sum_{i \geq 1} |\lambda_i|^2 - \left(\sum_{i \geq 1} |\lambda_i| \right)^2 \right| \leq \alpha(n)(|\lambda_n| + |\lambda_1|)^2$$

But

$$\sum_{i \geq 1} |\lambda_i|^2 = |C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2$$

and $\varepsilon_S^C(G) \leq \sqrt{n \left(|C| + 2 \sum_{j < i} (d_j + d_i)^2 d_{ij}^2 \right)}$ then the above inequality will be

$$n \left(|C| + 2 \sum_{j < i} (d_i + d_j)^2 d_{ij}^2 \right) - (\varepsilon_S^C(G))^2 \leq \alpha(n)(|\lambda_n| + |\lambda_1|)^2$$

$$\sqrt{n \left(|C| + 2 \sum_{j < i} (d_i + d_j)^2 d_{ij}^2 \right) - \alpha(n)(|\lambda_n| + |\lambda_1|)^2} \leq \varepsilon_S^C(G)$$

$$\Rightarrow \varepsilon_S^C(G) \geq \sqrt{n \left(|C| + 2 \sum_{j < i} (d_i + d_j)^2 d_{ij}^2 \right) - \alpha(n)(|\lambda_n| + |\lambda_1|)^2}$$

Theorem 5. Let $G = (n, m)$ be a simple graph. Let $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| > 0$ be a non-incremental order of minimum covering eigenvalues of $A_S^C(G)$ then

$$\varepsilon_S^C(G) \geq \frac{|C| + 2 \sum_{i < j} (d_i + d_j)^2 d_{ij}^2 + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}$$

Proof: The following inequality holds [12] if $R\mathbf{a}_i \geq \mathbf{b}_i \geq r\mathbf{a}_i R$ where $r, R, \mathbf{a}_i \neq \mathbf{0}$ and $\mathbf{b}_i$ are real numbers.

$$(R + r) \left(\sum_{i \geq 1}^n \mathbf{a}_i \mathbf{b}_i \right) \geq \sum_{i \geq 1}^n \mathbf{b}_i^2 + rR \sum_{i \geq 1}^n \mathbf{a}_i$$

Put $r = |\lambda_n|$, $\mathbf{a}_i = 1$, $R = |\lambda_1|$ and $\mathbf{b}_i = |\lambda_i|$ then

$$(|\lambda_1| + |\lambda_n|) \left(\sum_{i \geq 1}^n |\lambda_i| \right) \geq \sum_{i \geq 1}^n |\lambda_i|^2 + |\lambda_n||\lambda_1| \left(\sum_{i \geq 1}^n 1 \right)$$

$$|C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 + n|\lambda_n||\lambda_1| \leq (|\lambda_n| + |\lambda_1|) \varepsilon_S^C(G)$$

$$\therefore \varepsilon_S^C(G) \geq \frac{|C| + 2 \sum_{j < i} (d_i + d_j)^2 d_{ij}^2 + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}$$

In their papers [2] Bapat and S. Patil answers for the question of written of when does the graph energy becomes a rational number. In the same way the following theorem is obtained from the result of minimum covering schultz energy.

Theorem 6. If the minimum covering schultz energy $\varepsilon_S^C(G)$ is a rational number, then

$$\varepsilon_S^C(G) \equiv |C| \pmod{2}$$

Proof: Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the minimum covering schultz matrix $A_S^C(G)$ of a graph G of which $\lambda_1, \lambda_2, \dots, \lambda_r$ are positive and the rest of them are non-positive then

$$\begin{aligned} \sum_{i=1}^n |\lambda_i| &= (\lambda_1 + \lambda_2 + \dots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \dots + \lambda_n) \\ &= 2(\lambda_1 + \lambda_2 + \dots + \lambda_r) - (\lambda_1 + \lambda_2 + \dots + \lambda_n) \\ &= 2(\lambda_1 + \lambda_2 + \dots + \lambda_r) - \sum_{i=1}^n |\lambda_i| \\ &= 2(\lambda_1 + \lambda_2 + \dots + \lambda_r) - |C| \\ &\equiv |C| \pmod{2} \\ &\Rightarrow \sum_{i=1}^n |\lambda_i| \equiv |C| \pmod{2} \end{aligned}$$

$\varepsilon_S^C(G)$ FOR VARIOUS STANDARD GRAPHS

Theorem 1. For a complete graph K_n where $n \geq 2$

$$\varepsilon_S^C(K_n) = (2n^2 - 7n + 6) + \sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}$$

Proof: Let K_n be a complete graph having the vertex set $V(K_n) = \{v_1, v_2, \dots, v_n\}$ where the minimum vertex cover is $(n - 1)$. The minimum covering schultz set is $C = \{v_1, v_2, \dots, v_{n-1}\}$. The associated schultz matrix $A_S^C(K_n)$ is as follows:

$$A_S^C(K_n) = \begin{pmatrix} 1 & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1) & 1 & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1) & 2(n-1) & 1 & \dots & 2(n-1) & 2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 1 & 2(n-1) \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 0 \end{pmatrix}_{n \times n}$$

The characteristic polynomial of the schultz matrix $A_S^C(K_n)$ is

$$f_n(K_n, \lambda) = (-1)^n (\lambda + (2n - 3))^{n-2} (\lambda^2 - (2n^2 - 6n + 5)\lambda - (4n^3 - 12n^2 + 12n - 4)) = 0$$

The eigenvalues are $\lambda = -(2n - 3)$ with multiplicity $(n - 2)$, $\lambda = \frac{(2n^2 - 6n + 5) \pm \sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}}{2}$ with multiplicity one each.

The spectrum of $\text{Spec}_S^C(K_n)$ is

$$\text{Spec}_S^C(K_n) = \left(\frac{(2n^2 - 6n + 5) + \sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}}{2}, -2n + 3, \frac{(2n^2 - 6n + 5) - \sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}}{2} \right)$$

$\frac{1}{n - 2}$
 $\frac{1}{2}$

The minimum covering schultz energy of the complete graph K_n is

$$\begin{aligned} \varepsilon_S^C(K_n) &= \left| \frac{(2n^2 - 6n + 5)}{2} + \frac{\sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}}{2} \right| (1) + |-2n + 3|(-2 + n) \\ &\quad + \left| \frac{(2n^2 - 6n + 5)}{2} - \frac{\sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}}{2} \right| (1) \\ \varepsilon_S^C(K_n) &= \frac{(2n^2 - 6n + 5)}{2} + \frac{\sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}}{2} + (2n - 3)(n - 2) \\ &\quad - \frac{(2n^2 - 6n + 5)}{2} + \frac{\sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9}}{2} \\ \therefore \varepsilon_S^C(K_n) &= (2n^2 - 7n + 6) + \sqrt{4n^4 - 8n^3 + 8n^2 - 12n + 9} \end{aligned}$$

Theorem 2. For a cocktail party graph $K_{n \times 2}$ where $n \geq 2$

$$\varepsilon_S^C(K_{n \times 2}) = 16n^2 - 16n$$

Proof: Let $K_{n \times 2}$ be a cocktail party graph having the vertex set $V(K_{n \times 2}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where the minimum vertex cover is $(2n - 2)$. The minimum covering schultz set is $C = \{u_1, u_2, \dots, u_{n-1}, v_1, v_2, \dots, v_{n-1}\}$. The associated schultz matrix $A_S^C(K_{n \times 2})$ is as follows:

$$A_S^C(K_{n \times 2}) = \begin{pmatrix} 1 & (4n-4) & \dots & (4n-4) & (4n-4) & (8n-8) & (4n-4) & \dots & (4n-4) & (4n-4) \\ (4n-4) & 1 & \dots & (4n-4) & (4n-4) & (4n-4) & (8n-8) & \dots & (4n-4) & (4n-4) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (4n-4) & (4n-4) & \dots & 0 & (4n-4) & (4n-4) & (4n-4) & \dots & (8n-8) & (4n-4) \\ (4n-4) & (4n-4) & \dots & (4n-4) & 1 & (4n-4) & (4n-4) & \dots & (4n-4) & (8n-8) \\ (8n-8) & (4n-4) & \dots & (4n-4) & (4n-4) & 1 & (4n-4) & \dots & (4n-4) & (4n-4) \\ (4n-4) & (8n-8) & \dots & (4n-4) & (4n-4) & (4n-4) & 1 & \dots & (4n-4) & (4n-4) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (4n-4) & (4n-4) & \dots & (8n-8) & (4n-4) & (4n-4) & (4n-4) & \dots & 1 & (4n-4) \\ (4n-4) & (4n-4) & \dots & (4n-4) & (8n-8) & (4n-4) & (4n-4) & \dots & (4n-4) & 0 \end{pmatrix}_{2n \times 2n}$$

The characteristic polynomial of the schultz matrix $A_S^C(K_{n \times 2})$ is

$$f_n(K_{n \times 2}, \lambda) = (\lambda - 1)^{n-2} (\lambda + (8n - 9))^{n-1} (\lambda + (8n - 8)) (\lambda^2 - (8n^2 - 8n + 1)\lambda + (8n - 8)) = 0$$

The eigenvalues are $\lambda = 1$ with multiplicity $(n - 2)$, $\lambda = -(8n - 9)$ with multiplicity $(n - 1)$, $\lambda = -(8n - 8)$ with multiplicity 1 and $\lambda = \frac{(8n^2 - 8n + 1) \pm \sqrt{64n^4 - 128n^3 + 80n^2 - 48n + 33}}{2}$ with multiplicity one each.

The spectrum $Spec_S^C(K_{n \times 2})$ is

$$Spec_S^C(K_{n \times 2}) = \begin{pmatrix} -8n + 9 & 1 & \frac{X - \sqrt{Y}}{2} & \frac{X + \sqrt{Y}}{2} & -8n + 8 \\ n - 1 & n - 2 & 1 & 1 & 1 \end{pmatrix}$$

where $X = 8n^2 - 8n + 1$ and $Y = 64n^4 - 128n^3 + 80n^2 - 48n + 33$

The minimum covering schultz energy of $K_{n \times 2}$ is

$$\begin{aligned} \varepsilon_S^C(K_{n \times 2}) &= |-8n + 9|(n - 1) + |1|(n - 2) \\ &+ \left| \frac{(8n^2 - 8n + 1)}{2} + \frac{\sqrt{64n^4 - 128n^3 + 80n^2 - 48n + 33}}{2} \right| (1) \\ &+ \left| \frac{(8n^2 - 8n + 1)}{2} - \frac{\sqrt{64n^4 - 128n^3 + 80n^2 - 48n + 33}}{2} \right| (1) + |-8n + 8|(1) \end{aligned}$$

$$\begin{aligned} \varepsilon_S^C(K_{n \times 2}) &= (8n - 9)(n - 1) + (n - 2) + \frac{(8n^2 - 8n + 1)}{2} \\ &+ \frac{\sqrt{64n^4 - 128n^3 + 80n^2 - 48n + 33}}{2} + \frac{(8n^2 - 8n + 1)}{2} \\ &- \frac{\sqrt{64n^4 - 128n^3 + 80n^2 - 48n + 33}}{2} + (8n - 8) \end{aligned}$$

$$\varepsilon_S^C(K_{n \times 2}) = (8n^2 - 17n + 9) + (n - 2) + (8n^2 - 8n + 1) + (8n - 8)$$

$$\therefore \varepsilon_S^C(K_{n \times 2}) = 16n^2 - 16n$$

Theorem 3. For star graph $K_{1,n-1}$ where $n \geq 3$

$$\varepsilon_S^C(K_{1,n-1}) = (4n - 8) + \sqrt{4n^3 + 12n^2 - 72n + 81}$$

Proof: Let $K_{1,n-1}$ be a star graph having the vertex set $V(K_{1,n-1}) = \{v_0, v_1, v_2, \dots, v_n\}$ where the minimum vertex cover is 1. The minimum covering schultz set is $C = \{v_0\}$. The associated schultz matrix $A_S^C(K_{1,n-1})$ is as follows:

$$A_S^C(K_{1,n-1}) = \begin{pmatrix} 1 & n & n & n & \dots & n \\ n & 0 & 4 & 4 & \dots & 4 \\ n & 4 & 0 & 4 & \dots & 4 \\ n & 4 & 4 & 0 & \dots & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 4 & 4 & 4 & \dots & 0 \end{pmatrix}_{(n+1) \times (n+1)}$$

The characteristic polynomial of $A_S^C(K_{1,n-1})$ is

$$f_n(K_{1,n-1}, \lambda) = (-1)^{n-2} (\lambda + 4)^{n-2} (\lambda^2 - (4n-7)\lambda - (n^3 - n^2 - 4n + 8)) = 0$$

The eigenvalues are $\lambda = -4$ with multiplicity $(n-2)$ and $\lambda = \frac{(4n-7) \pm \sqrt{4n^3 + 12n^2 - 72n + 81}}{2}$ with multiplicity one each.

The spectrum $Spec_S^C(K_{1,n-1})$ is

$$Spec_S^C(K_{1,n-1}) = \left(\begin{array}{ccc} \frac{(4n-7) + \sqrt{4n^3 + 12n^2 - 72n + 81}}{2} & -4 & \frac{(4n-7) - \sqrt{4n^3 + 12n^2 - 72n + 81}}{2} \\ 1 & n-2 & 1 \end{array} \right)$$

The minimum covering Schultz energy of $K_{1,n-1}$ is

$$\varepsilon_S^C(K_{1,n-1}) = \left| \frac{(4n-7) + \sqrt{4n^3 + 12n^2 - 72n + 81}}{2} \right| (1) + |-4|(n-2) + \left| \frac{(4n-7) - \sqrt{4n^3 + 12n^2 - 72n + 81}}{2} \right| (1)$$

$$\varepsilon_S^C(K_{1,n-1}) = \frac{(4n-7)}{2} + \frac{\sqrt{4n^3 + 12n^2 - 72n + 81}}{2} + 4(n-2) - \frac{(4n-7)}{2} + \frac{\sqrt{4n^3 + 12n^2 - 72n + 81}}{2}$$

$$\therefore \varepsilon_S^C(K_{1,n-1}) = (4n-8) + \sqrt{4n^3 + 12n^2 - 72n + 81}$$

Theorem 4. For a complete bipartite graph $K_{m,n}$ where $m \leq n$

$$\varepsilon_S^C(K_{m,n}) = 16mn - 9m - 8n + 9$$

Proof: Let $K_{m,n}$ be a complete bipartite graph having the vertex set $V(K_{m,n}) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$ where the minimum vertex cover is $\min(m, n)$. The associated schultz matrix $A_S^C(K_{m,n})$ is as follows:

$$A_S^C(K_{m,n}) = \begin{pmatrix} 1 & 4n & 4n & \dots & 4n & (m+n) & (m+n) & (m+n) & \dots & (m+n) \\ 4n & 1 & 4n & \dots & 4n & (m+n) & (m+n) & (m+n) & \dots & (m+n) \\ 4n & 4n & 1 & \dots & 4n & (m+n) & (m+n) & (m+n) & \dots & (m+n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 4n & 4n & 4n & \dots & 1 & (m+n) & (m+n) & (m+n) & \dots & (m+n) \\ (m+n) & (m+n) & (m+n) & \dots & (m+n) & 0 & 4m & 4m & \dots & 4m \\ (m+n) & (m+n) & (m+n) & \dots & (m+n) & 4m & 0 & 4m & \dots & 4m \\ (m+n) & (m+n) & (m+n) & \dots & (m+n) & 4m & 4m & 0 & \dots & 4m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (m+n) & (m+n) & (m+n) & \dots & (m+n) & 4m & 4m & 4m & \dots & 0 \end{pmatrix}_{2m \times 2n}$$

The characteristic polynomial of $A_S^C(K_{m,n})$ is

$$f_n(K_{m,n}, \lambda) = (-1)^{m-2+n} (\lambda + 4m)^{-1+n} (\lambda + (4n-1))^{-1+m} (\lambda^2 - (8mn - 4m - 4n + 8)\lambda + (mn^3 - 14m^2n^2 + 16mn^2 + m^3n + 16m^2n - 20)) = 0$$

The eigenvalues are $\lambda = -4m$ with multiplicity $(n - 1)$, $\lambda = -(4n - 1)$ with multiplicity $(m - 1)$ and $\lambda = \frac{(4mn-2n-2m+4) \pm \sqrt{mn^3+(2m^2+4)n^2+(m^3+20m-16)n+(4m^2-12m+16)}}{2}$ with multiplicity one each.

The spectrum $Spec_S^C(K_{m,n})$ is

$$Spec_S^C(K_{m,n}) = \begin{pmatrix} \sqrt{Y} + X & \sqrt{Y} - X & -4m & -4n + 1 \\ 1 & 1 & n - 1 & m - 1 \end{pmatrix}$$

where $X = 4mn - 2n - 2m + 4$ and $Y = mn^3 + (2m^2 + 4)n^2 + (m^3 + 20m - 16)n + 4m^2 - 12m + 16$

The minimum covering schultz energy of $\epsilon_S^C(K_{m,n})$ is

$$\epsilon_S^C(K_{m,n}) = |\sqrt{Y} + X|(1) + |\sqrt{Y} - X|(1) + |-4m|(n - 1) + |-4n + 1|(m - 1)$$

$$\epsilon_S^C(K_{m,n}) = \sqrt{Y} + X - \sqrt{Y} + X + 4m(n - 1) + (4n - 1)(m - 1)$$

$$\epsilon_S^C(K_{m,n}) = 2X + 4mn - 4m + 4mn - m - 4n + 1$$

$$\epsilon_S^C(K_{m,n}) = 2(4mn - 2n - 2m + 4) + 4mn - 4m + 4mn - m - 4n + 1$$

$$\therefore \epsilon_S^C(K_{m,n}) = 16mn - 9m - 8n + 9$$

CONCLUSION

The study focuses on the minimum covering schultz energy of a graph and its importance in analyzing the structural efficiency and coverage of complex networks. It introduces the minimum covering Schultz matrix a new degree-based matrix representation that incorporates covering properties enabling refined spectral analysis with reduced computational complexity. A new spectral invariant called the minimum covering schultz energy is defined using this matrix extending the concept of classical graph energy. The research paper establishes theoretical bounds and computes exact values for standard graph families strengthening the algebraic understanding of the relationship between covering and degree of a graph. Additionally, schultz energy is highlighted for its applications in chemical graph theory and network analysis as it reflects global structural characteristics of graphs. Current research focuses on determining bounds identifying extremal graphs exploring relationships with other graph invariants and characterizing graphs with equal covering energy.

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