

A Comparative Analysis of Fixed Point Results in Banach and Hilbert Spaces

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ABSTRACT

The main objective of this paper is to use contraction mapping and non-expansive operators for unique fixed point results in real Banach and Hilbert spaces. A fundamental role of the fixed point theory plays in optimization and nonlinear functional analysis. This work is a generalization of the unique fixed point and its discussion. We have collected different ideas for unique fixed points in real Banach and Hilbert spaces. Also, we formulate and compare fixed point conditions involving contractive inequalities, the role of completeness, norm structure, inner product geometry, lower semicontinuous auxiliary function and Hausdorff distance for set valued maps. Finally, we have conducted a comparative study and analysis of unique fixed points in Banach and Hilbert spaces. New comparative study, observation, differences analysis and implementation of these structure features are also provided.

Keywords- Normed space, Banach space, Hilbert space, inner product, unique fixed point, contraction mapping, non-expansive mapping

INTRODUCTION

Functional analysis, a fundamental branch of mathematical analysis, primarily studies vector spaces with topological structures that allow the study of convergence, continuity and linear operators, among the most significant types of vector spaces in functional analysis are Banach spaces and Hilbert spaces. While both are complete normed spaces, they have key differences, particularly in their structure and properties. Banach spaces are general complete normed vector spaces, whereas Hilbert spaces are a specific subclass of Banach spaces that also possess an inner product. This work is motivated by real Banach and Hilbert spaces for unique fixed point results and comparative analysis. The aim of this special issue is to collect the related results for unique fixed points in real Banach spaces and Hilbert spaces. Fixed points theory have applications in nonlinear functional analysis, integral equation and differential, optimization, dynamical systems and approximation theory.

The fixed point theorem induced by Stefan Banach, [1] was first declared by in the year 1922 and later by kennan [2]. The contraction mapping theorem was developed through the work of meir and keeler, [3] and Chatterjea, [4]. The fixed point theory and the contemporary extension the extension of Ekeland and Reich, [5, 6]. The fixed point theorem for contractive mapping in complete G- metric space was established through the work done of Mustafa and Sims, [7]. Some applications of Caristi's fixed point theorem were introduced by Farshid, Erdal, and Hassan, [8] in 2016. The fixed point theorem and convergence theorem for some generalized nonexpansive mappings were developed by done of Suzuki, [9]. The generalized metric space and mapping with fixed points were studied by Dhage, [10] in 1992. A common fixed point theorem in metric space was extended by Sedge, Shobe, and Zhou, [11] in 2007. A nonexpansive type mapping and fixed points to the development of new ideas by Ciric, [12] in 1993. A common fixed point of mapping in uniformly convex spaces was introduced by Bose, [13]. The weak convergence theorem for nonexpansive mapping in Banach space was developed by Reach, and Kr., [14]. The fixed point theory and its application were studied by Srivastav, [15]. A non-expansive mapping and fixed point in Banach spaces were developed

by work of Belluce, and Kirk, [16]. The nonexpansive mapping and Fixed Point in Banach spaces study by Belluce, and Kirk, [17] in 1969. The nonlinear functional analysis and its application was developed by Zeidler, [18] in 1986. The present work focuses on fixed point results in Banach and Hilbert Spaces.

The novelty of this work lies in

- Constructing unique fixed-point results under various contractive conditions.
- Systematic comparative treatment and fixed point condition implementation within real Banach space and Hilbert space.

This approach generalizes classical Banach, Kannan, Chatterjea and Reich contractions within a specific subclass structure.

Preliminaries

Basic definition

Fixed point

Suppose $(\mathcal{M}, d)$ be a real metric space with a mapping $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$ in which a point β belongs to $\mathcal{M}$. Then a point β belongs to $\mathcal{M}$ is called a fixed point of $\mathcal{F}$ if β is mapped into itself. That is a $\mathcal{F}(\beta) = \beta$, for some β

EX: (1) The function $\mathcal{F}: \mathbb{R} \rightarrow \mathbb{R}$ satisfied as $\mathcal{F}(\beta) = 2$ and has a fixed point.

Hence, 0 is a fixed point.

(2) The function $\mathcal{F}: \mathbb{R} \rightarrow \mathbb{R}$ satisfied as satisfies $\mathcal{F}(\beta) = \beta^2$ and has two fixed points.

Clearly, 0 and 1 are fixed points.

(3) The function $\mathcal{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $\mathcal{F}(\beta, \eta) = \beta$ and has infinite fixed points.

Normed Space

Assume $\mathcal{M}$ is a linear space (LS) over a field $\mathcal{F}$ and a function $\| \cdot \|: \mathcal{M} \rightarrow \mathbb{R}$ is defined such that

(1) $\| \beta \| \geq 0$, $\| \beta \| = 0$, if and only if $\beta = 0$, for all $\beta \in \mathcal{M}$.

(2) $\| \beta + \eta \| \leq \| \beta \| + \| \eta \|$, for all $\beta, \eta \in \mathcal{M}$.

(3) $\| \mathcal{K}\beta \| = |\mathcal{K}| \| \beta \|$, for all $\beta \in \mathcal{M}$, $\mathcal{K} \in \mathcal{F}$.

Then $\| \cdot \|$ is called a norm on $\mathcal{M}$, and the structure $(\mathcal{M}, \| \cdot \|)$ is called a real normed space, and every convergent Cauchy sequence in a real **normed space** is satisfied and every convergent Cauchy Sequence of real normed space is satisfied, then that is called a real **Banach space**.

Inner product space

Let $\mathcal{M}$ be a linear space over a field k ($\mathbb{R}$ or $\mathbb{C}$). A function $\mathcal{M} \times \mathcal{M} \rightarrow K$ that assigns each ordered pair $(\beta, \eta) \in \mathcal{M} \times \mathcal{M}$ to a unique scalar $\beta, \eta \in K$ is said to be an Inner Product if it satisfies the following conditions for all $\beta, \eta, \mu \in \mathcal{M}$ and $\rho \in k$.

Inner Product Space

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- (1) $\langle \beta, \eta \rangle = \overline{\langle \eta, \beta \rangle}$ (conjugate symmetric)
- (2) $\langle \beta + \mu, \eta \rangle = \langle \beta, \eta \rangle + \langle \mu, \eta \rangle$ (Additivity)
- (3) $\rho \langle \beta, \eta \rangle = \langle \rho\beta, \eta \rangle$ (Homogeneity)
- (4) $\langle \beta, \beta \rangle = 0$ and $\langle \beta, \beta \rangle = 0$ iff $\beta = 0$ (positivity)

Hilbert space

A complete ($\mathbb{R}$ or $\mathbb{C}$) inner product space is called a Hilbert space,

Then the form,

$$\|x\| = \sqrt{\langle x, x \rangle}$$

➤ Assume a mapping $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$, in which $(\mathcal{M}, \|\cdot\|)$ is a real normed space.

Where $\rho: \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is a mapping that is lower Semi-continuous (L.S) with respect to the first variable w. r. t. first variable,

$$\|\beta - \eta\| \leq \rho(\beta, \eta) - \rho(\mathcal{F}\beta, \mathcal{F}\eta), \forall \beta, \eta \in \mathcal{M}.$$

Then the point has a fixed point and uniqueness of $\mathcal{F}$.

➤ Let us assume a map $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$ in which $(\mathcal{M}, \|\cdot\|)$ is a real normed space

Such as that,

$$\|\beta - \mathcal{F}\beta\| \leq \rho(\beta) - \rho(\mathcal{F}\beta), \forall \beta \in \mathcal{M}. \text{ Where } \rho: \mathcal{M} \rightarrow [0, \infty) \text{ is a mapping of}$$

Lower Semicontinuous (L.S) and a fixed point in which N is the natural number with R being the real numbers.

➤ Let a mapping $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$, and assume $(\mathcal{M}, \|\cdot\|)$ is a real normed space.

That is,

$$\|\mathcal{F}\beta - \mathcal{F}\eta\| \leq |\mathcal{K}| \|\beta - \eta\|, \forall \beta, \eta \in \mathcal{M}. \text{ Where } \mathcal{K}: [0, \infty) \rightarrow [0, \infty) \text{ is a mapping of}$$

Lower Semicontinuous (L.S) (t) $<$, for each $t > 0$ and $(t)/t$ is a mapping of non-decreasing.

Then the point has a fixed point and uniqueness.

Main Results

In this paper, we show a comparative analysis of Banach and Hilbert space.

Uniqueness in Banach space

Theorem: Fixed Point Through a Lower Semi-continuous Function

Let a mapping $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$ and $(\mathcal{M}, \|\cdot\|)$ be a real normed space. Where $\rho: \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$.

Such that,

$$\|\beta - \eta\| \leq \rho(\beta, \eta) - \rho(\mathcal{F}\beta, \mathcal{F}\eta), \forall \beta, \eta \in \mathcal{M} \dots \quad (1)$$

Then, $\mathcal{F}$ has a unique of fixed point.

Proof:

Since $(\mathcal{M}, \|\cdot\|)$ be a real normed space.

$$\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$$

Where $\rho: \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$.

Define the function $\Psi: \mathcal{M} \rightarrow [0, \infty)$

$$\eta = \mathcal{F}\beta$$

$$\|\beta - \mathcal{F}\beta\| \leq \rho(\beta, \mathcal{F}\beta) - \rho(\mathcal{F}\beta, \mathcal{F}^2\beta)$$

Since ρ is lower semi-continuous with respect to its first variable, the function Ψ required lower Semi-continuous property under the corresponding assumption on $\mathcal{F}$.

Now suppose that $\mathcal{F}$ has a fixed point $\beta^* \in \mathcal{M}$.

Then

$$\Psi(\beta^*) = \rho(\beta^*, \mathcal{F}\beta^*) = \rho(\beta^*, \beta^*),$$

Now from equation (1)

Suppose that β^* and η^* are two fixed point of $\mathcal{F}$.

$$\text{Thus, } \mathcal{F}\beta^* = \beta^*, \quad \mathcal{F}\eta^* = \eta^*.$$

$$\|\beta^* - \eta^*\| \leq \rho(\beta^*, \eta^*) - \rho(\mathcal{F}\beta^*, \mathcal{F}\eta^*),$$

$$= \rho(\beta^*, \eta^*) - \rho(\beta^*, \eta^*),$$

$$= 0$$

$$, \quad \beta^* = \eta^*,$$

Hence, the fixed point of $\mathcal{F}$ is unique.

Theorem: Banach Contraction theory

Let $(\mathcal{M}, \|\cdot\|)$ be a real Banach space and let a mapping $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$, for some $\mathcal{K} \in [0, 1)$

When,

$$\| \mathcal{A}\beta - \mathcal{F}\eta \| \leq |\mathcal{K}| \| \beta - \eta \|, \quad \forall \beta, \eta \in \mathcal{M}.$$

Proof: Since $(\mathcal{M}, \| \cdot \|)$ be a real Banach space and

$$\| \mathcal{A}\beta - \mathcal{F}\eta \| \leq |\mathcal{K}| \| \beta - \eta \|, \quad \forall \beta, \eta \in \mathcal{M} \text{ and } \in [0, 1)$$

Choose an arbitrarily $\forall \beta_0 \in \mathcal{M}$ and define the iterative sequence

$$\beta_{n+1} = \mathcal{F}\beta_n, \quad n = 0, 1, 2 \dots$$

$$\| \beta_{n+1} - \beta_n \| = \| \mathcal{A}\beta_n - \mathcal{F}\beta_{n-1} \| \leq \mathcal{K} \| \beta_n - \mathcal{A}\beta_{n-1} \|$$

$$\| \beta_{n+1} - \beta_n \| \leq \mathcal{K}^n \| \beta_1 - \beta_0 \|$$

For $m > n$ by the triangle inequality

$$\begin{aligned} \| \beta_m - \beta_n \| &\leq \sum_{i=n}^{m-1} \| \beta_{i+1} - \beta_i \| \\ &\leq \sum_{i=n}^{m-1} \mathcal{K}^i \| \beta_1 - \beta_0 \| \\ &\leq \frac{\mathcal{K}^n}{1-\mathcal{K}} \| \beta_1 - \beta_0 \| \end{aligned}$$

$$\frac{\mathcal{K}^n}{1-\mathcal{K}} \| \beta_1 - \beta_0 \| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\beta_n \rightarrow \beta^*$$

Since $\mathcal{F}$ is Lipschitz continuous,

$$\| \mathcal{A}\beta_n - \mathcal{F}\beta^* \| \leq \mathcal{K} \| \beta_n - \beta^* \| \rightarrow 0$$

$$\mathcal{A}\beta_n \rightarrow \mathcal{F}\beta^*$$

$$\beta_{n+1} \rightarrow \beta^*$$

$$\mathcal{A}\beta^* = \beta^*$$

To prove uniqueness, suppose that γ^* is another point of $\mathcal{F}$. Then

$$\mathcal{A}\beta^* = \beta^*, \quad \mathcal{F}\gamma^* = \gamma^*,$$

$$\| \mathcal{A}\beta^* - \gamma^* \| = \| \mathcal{A}\beta^* - \mathcal{F}\gamma^* \|$$

$$\leq \mathcal{K} \| \beta^* - \gamma^* \|$$

$$(1 - \mathcal{K}) \| \beta^* - \gamma^* \| \leq 0$$

$$\| \beta^* - \gamma^* \| = 0$$

$$\beta^* = \gamma^*$$

Hence, the uniqueness fixed point in $\mathcal{M}$.

Theorem: Multivalued Fixed Point Result

Let $(\mathcal{M}, \|\cdot\|)$ is a real Banach space and let the family of all non- empty closed bounded subsets of $\mathcal{M}$, denoted by $\mathcal{CB}(\mathcal{M})$, is a contraction mapping $\mathcal{F}:\mathcal{M}\rightarrow\mathcal{CB}(\mathcal{M})$. Suppose that there exist

$$\mathcal{K} \in [0, 1) \text{ such that } H(\mathcal{F}\beta - \mathcal{F}\eta) \leq \mathcal{K} \|\beta - \eta\|, \forall \beta, \eta \in \mathcal{M}, \beta^* \in \mathcal{F}\beta^*$$

Proof:

Now we first choose an arbitrarily $\beta_0 \in \mathcal{M}$

Since,

$$\mathcal{F}\beta_0 \neq \emptyset$$

$$\beta_1 \in \mathcal{F}\beta_0$$

Here, using the Hausdorff contraction condition,

Then, we can one induction choose

$$\beta_{n+1} \in \mathcal{F}\beta_n$$

$$\|\beta_{n+1} - \beta_n\| \leq \mathcal{K} \|\beta_n - \beta_{n-1}\|, n \geq 1.$$

$$\|\beta_{n+1} - \beta_n\| \leq \mathcal{K}^n \|\beta_1 - \beta_0\|$$

For $m > n$

$$\begin{aligned} \|\beta_m - \beta_n\| &\leq \sum_{i=n}^{m-1} \|\beta_{i+1} - \beta_i\| \\ &\leq \sum_{i=n}^{m-1} \mathcal{K}^i \|\beta_1 - \beta_0\| \\ &\leq \frac{\mathcal{K}^n}{1-\mathcal{K}} \|\beta_1 - \beta_0\| \end{aligned}$$

$$\|\beta_1 - \beta_0\| \rightarrow 0 \text{ as, } m, n \rightarrow \infty$$

$$\beta_n \rightarrow \beta^*$$

Since $\mathcal{M}$ is a Banach space, there exist, $\beta^* \in \mathcal{M}$

Such that,

$$\beta_n \rightarrow \beta^*$$

it remains to show that

$$\beta^* \in \mathcal{F}\beta^*$$

$$H(\mathcal{F}\beta_n, \mathcal{F}\beta^*) \leq \mathcal{K} \|\beta_n - \beta^*\| \rightarrow 0$$

$$\beta_{n+1} \in \mathcal{F}\beta_n$$

$$\beta^* \in \mathcal{F}\beta^*$$

Hence $\mathcal{F}$ has a fixed point in $\mathcal{M}$.

Uniqueness in Hilbert Space

Theorem

Let H is a real Hilbert space and let $\mathcal{F}: H \rightarrow H$

$$\|\mathcal{F}\beta - \mathcal{F}\eta\| \leq \mathcal{K} \|\beta - \eta\|, \forall, \beta, \eta \in H$$

Proof:

Since every Hilbert space is complete with respect to the norm induced by its inner product

$$\|\beta\| = \sqrt{(\beta, \beta)},$$

The given inequality shows that $\mathcal{F}$ is a contraction by contraction theory,

there exist $\beta^* \in H$ such that $\mathcal{F}\beta^* = \beta^*$.

To prove uniqueness, suppose that $\eta^* \in H$ is another fixed point .

Then $\mathcal{F}\beta^* = \beta^*, \mathcal{F}\eta^* = \eta^*$.

$$\|\beta^* - \eta^*\| = \|\mathcal{F}\beta^* - \mathcal{F}\eta^*\|$$

$$\leq \mathcal{K} \|\beta^* - \eta^*\|$$

$$(1 - \mathcal{K}) \|\beta^* - \eta^*\| \leq 0$$

$$\beta^* = \eta^*$$

Therefore, $\mathcal{F}$ has a unique fixed point in H .

Differences Between Banach and Hilbert Spaces

Property	Banach Space	Hilbert Space
Structure	Normed vector space	Inner product space (a Banach space with an inner product)
Inner Product	Not necessarily defined	Always defined
Geometry	Lacks an inherent geometric structure	Has rich geometric properties like angles, orthogonality, and projections
Orthogonality	Not well-defined	Well-defined using the inner product
Best Approximation	May not hold	Always holds due to orthogonality principle
Non-expansive Mapping	May fail to have fixed points	Always have fixed point inclosed

Comparative Analysis

Banach and Hilbert spaces are both fundamental settings in functional analysis. A Banach space is a complete normed vector space, whereas a Hilbert space is complete inner-product space.

Fixed point theorems concentrate on the real sequences and the stability of unique fixed (UF) points under repeated applications of real numbers. Uniqueness theorems aim to prove the exclusive and singular solutions. Fixed point theorems are more frequently used in the study of real Banach space. Uniqueness theorems are often concerned with differential equations. Banach space and Hilbert space are central in modern analysis, with Banach space providing a more general framework, while Hilbert space offers stronger geometric and functional properties. Understanding their distinctions and connections is crucial for fields like mathematical physics, optimization, and PDE theory.

CONCLUSION

In this paper, we have presented the generalization of the fixed point theorem generalization in real Banach space and successfully obtained the results of the unique fixed (UF) point. We have constructed some differentiated types of examples to assist with contraction mapping in real Banach space and Hilbert space. The nonlinear equation has to be verified with a unique fixed (UF) point and comparative analysis, such as the condition of Banach space and Hilbert space. Since every Hilbert space is a Banach space, the same principle applies in Hilbert space.

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Conflicts of Interest

The author declares that he has no conflicts of interest

Institutional Review Board Statement

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