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Fuzzy Lucky Labeling and Proper Fuzzy Lucky Labeling of Special Graphs

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Abstract—This paper introduces fuzzy lucky labelling and proper fuzzy lucky labeling of graphs. A fuzzy lucky labeling assigns fuzzy values from (0, 1] to vertices such that induced sums at adjacent vertices differ, while the proper version additionally requires distinct labels for neighbours. We establish key theorems, including existence results and relationships with chromatic number, and compute fuzzy lucky numbers for special graphs such as stars, bull graphs, and fork graphs.

Index Terms—Fuzzy lucky labeling, proper labeling, special graphs, graph theory, fuzzy graphs.

I. Introduction

Graph labeling is a significant area of graph theory that assigns values (usually integers or weights) to graph elements (vertices or edges) according to certain rules. One such labeling is lucky labeling, where vertex labels are assigned so that the sum of neighbor labels distinguishes adjacent vertices. Extending this idea, fuzzy lucky labeling introduces fuzziness into the labeling process by assigning values from the continuous interval (0, 1].

In fuzzy lucky labeling, each vertex is assigned a fuzzy value, and the induced sum over neighbors ensures distinctiveness between adjacent vertices. A stronger version, proper fuzzy lucky labeling, requires both the induced sums and the vertex labels of adjacent vertices to be distinct. These concepts allow for deeper exploration of structural properties of graphs, particularly for special graph families such as star graphs, bull graphs, and fork graphs.

The motivation for fuzzy graph labeling arises from the increasing need to model real-world systems that exhibit uncertainty, vagueness, and imprecision. Applications range from communication networks to decision making systems where classical crisp labeling may not adequately capture complexities.

II. Literature Review

Graph labeling has evolved significantly over the past six decades, beginning with Rosa's (1960) graceful labelling and expanded through surveys such as Gallian's dynamic catalogue, which documents hundreds of results. Foundational studies, including Harary's sum graphs and Chen's integral sum graphs, established labeling as both a combinatorial and applied tool.

Proper labeling, introduced by Karo´nski et al. (2004), connected labeling with graph coloring and led to complexity analyses, notably by Dehghan et al. (2013), who proved NP-completeness for several variants. Later, Czerwi´nski, Grytczuk, and Łazny (2009) proposed lucky labeling, where neighbor sums differ across edges. This idea spurred extensive research: Ahadi et al. (2012) proved NP-hardness, while Akbari et al. (2013) explored chromatic links. Subsequent works extended lucky labelling to special graphs such as bloom, meshes, trees, and quadrilateral snakes, while proper lucky labeling refined the definition further.

Fuzzy lucky labeling, inspired by Zadeh's (1965) fuzzy set theory, generalizes labeling with fuzzy numbers. Early contributions by Kumar, Meenakshi, and others have applied it to jewel, comb, fan, and double star graphs, but the field remains in its infancy. In parallel, special graph families—snakes, ladders, and meshes have been central to testing labeling schemes, with results spanning graceful, product-cordial, quotient, and prime labelings. Related studies on edge and irregular labelings further broaden the landscape.

Applications of labeling extend to interdisciplinary domains such as chemical graph theory, domination problems, and network structures, underscoring its theoretical and applied importance. Overall, while significant progress has been achieved, gaps remain in extending lucky and fuzzy lucky labeling to broader graph classes and in developing constructive algorithms—directions that motivate the present research.

Definition II.1. Let G = (V,E) be a graph. A fuzzy lucky labeling of G is a function f : V (G) → (0, 1] that assigns a fuzzy value to each vertex. For each vertex u ∈ V (G), define

$$s_f(u) = \sum_{uv\epsilon E(G)} f(v).$$

If $s_f\,(u) \neq\; s_f\,(v)$ for every edge uv ∈ E(G), then f is called a fuzzy lucky labeling of G.

Definition II.2. Let G = (V,E) be a graph with a fuzzy lucky labeling f. A Proper Fuzzy Lucky Labeling of G is a fuzzy lucky labeling f such that $f(u) \neq f\,(v)$ for every edge uv ∈ E(G).

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In other words, in addition to the condition $s_f\ (u) \neq\ s_f\ (v)$, adjacent vertices must receive distinct fuzzy labels.

Definition II.3. Let G = (V,E) be a graph. A fuzzy lucky labeling of G is a function f : V (G) → (0, 1] such that for every edge uv ∈ E(G) we have $s_f\ (u) \neq\ s_f\ (v)$, where

$$s_f(u) = \sum_{uv \in E(G)} f(v).$$

The fuzzy lucky number of G, denoted by $\eta_f\ (G)$, is defined as $\eta_f\ (G) = \min\{|f(V(G))|$: f is a fuzzy lucky labeling of G} that is, the minimum cardinality of the set of fuzzy labels required to obtain a fuzzy lucky labeling of G.

Example II.1. Consider the star graph $K_{1,3}$ with center v_0 and leaves v_1, v_2, v_3.

Fuzzy Lucky Labeling: Assign fuzzy labels as

$$f(v_i)\ =\ 0.7, i\ =\ 0, 1, 2, 3.$$

For each vertex:

$$s_f\ (v_0)\ =\ f(v_1)\ +\ f(v_2)\ +\ f(v_3)$$
$$=\ 0.7\ +\ 0.7\ +\ 0.7\ =\ 2.1,$$
$$s_f\ (v_1)\ =\ f(v_0)\ =\ 0.7,$$
$$s_f\ (v_2)\ =\ f(v_0)\ =\ 0.7,$$
$$s_f\ (v_3)\ =\ f(v_0)\ =\ 0.7,$$

Here $s_f\ (v_0) \neq\ s_f\ (v_i)$ for i = 1, 2, 3. Hence f is a fuzzy lucky labeling.

Diagram for Fuzzy Lucky Labeling:

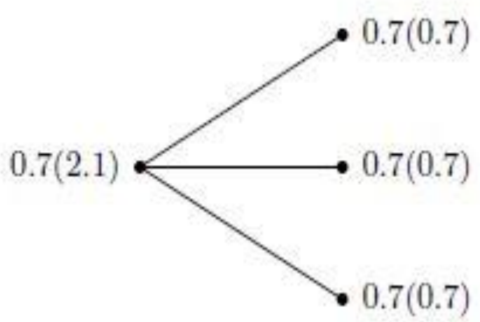

Fig. 1: FL Labeled $K_{1,3}$

Proper Fuzzy Lucky Labeling: If we require distinct fuzzy values for adjacent vertices, we can assign:

$$f(v_0) =\ 0.7, \qquad f(v_i) =\ 0.4, i\ =\ 1, 2, 3$$

For each vertex:

$$s_f\ (v_0)\ =\ f(v_1)\ +\ f(v_2)\ +\ f(v_3)$$
$$=\ 0.4 + 0.4 + 0.4\ = 1.2,$$
$$s_f\ (v_1)\ =\ f(v_0)\ =\ 0.7,$$
$$s_f\ (v_2)\ =\ f(v_0)\ =\ 0.7,$$
$$s_f\ (v_3)\ =\ f(v_0)\ =\ 0.7,$$

Here $s_f\ (v_0) \neq\ s_f\ (v_i)$ and also $f(v_0) \neq f(v_i)$ for all i = 1, 2, 3. Thus, f is a Proper Fuzzy Lucky Labeling.

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Diagram for Proper Fuzzy Lucky Labeling:

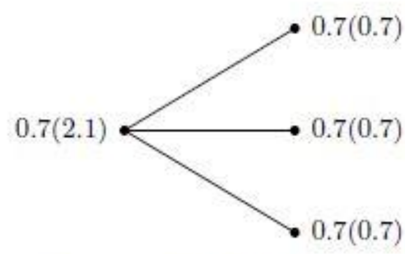

Fig. 1: FL Labeled $K_{1,3}$

III. Theorems

Theorem III.1. Every non-trivial connected graph ad mits a fuzzy lucky labeling.

Proof. Let G = (V, E) be a connected graph with n vertices. Assign to each vertex vi ∈ V (G) a distinct fuzzy value $f(v_i) = \frac{i}{n}$, where i = 1,2,...,n. Then for every edge uv ∈ E(G), the induced sums $s_f\ (u)$ and $s_f\ (v)$ involve different sets of fuzzy values, and hence cannot be equal. Thus f is a fuzzy lucky labeling of G.

Theorem III.2. Every Proper Fuzzy Lucky Labeling is a fuzzy lucky labeling, but not conversely.

Proof. By definition, a Proper Fuzzy Lucky Labeling satisfies two conditions: (i) $s_f\ (u) \neq\ s_f\ (v)$ for every uυ ∈ E(G), and (ii) $f(u) \neq\ f(v)$, for every uυ ∈ E(G). Condition (i) alone defines a fuzzy lucky labeling. Hence, every Proper Fuzzy Lucky Labeling is a fuzzy lucky labeling.

However, the converse does not hold: consider the star K1,3 with fuzzy labels $f(v_0) =\ 0.7,\ f(v_i) =\ 0.2, f(v_3) =\ 0.4$. This is a fuzzy lucky labeling since $s_f\ (v_0)\ =\ 0.8 \neq 0.7,$ but not a Proper Fuzzy Lucky Labeling because $f(v_1) =\ f(v_2)$. Thus, the converse is false.

Theorem III.3. Every tree admits a Proper Fuzzy Lucky Labeling.

Proof. Let T be a tree on n vertices. Since T is bipartite, let the bipartition be (X, Y). Assign distinct fuzzy values from [0,0.5] to the vertices of X and distinct fuzzy values from (0.5,1] to the vertices of Y. Thus adjacent vertices always receive distinct labels. Further, for any vertex u ∈ X, $s_f\ (u)$ is the sum of fuzzy values from (0.5,1], whereas for any vertex v ∈ Y , $s_f\ (v)$ is the sum of fuzzy values from [0,0.5]. Hence $s_f\ (u) \neq\ s_f\ (v)$ for adjacent vertices u, υ. Therefore, f is a Proper Fuzzy Lucky Labeling.

Theorem III.4. For any graph G, the proper fuzzy lucky number $\eta_f\ *$(G) is at least the chromatic number χ(G).

Proof. By definition, in a Proper Fuzzy Lucky Labeling adjacent vertices receive distinct fuzzy labels. Thus, the set of fuzzy labels defines a proper coloring of G. Therefore, the number of distinct fuzzy values used is at least χ(G). Hence, $\eta_f\ *$ (G) ≥ χ(G).

Theorem III.5. For a star graph $K_{1,n}$, the proper fuzzy lucky number is n +1.

Proof. In $K_{1,n}$, let υ_0 be the center and υ_1, υ 2,..., υ_n the leaves. Since υ_0 is adjacent to all leaves, $f(v_0)$ must differ from every $f(v_1)$. Also, the leaves vi are not adjacent to each other, so they may reuse fuzzy labels in a fuzzy lucky labeling. But in a Proper Fuzzy Lucky Labeling, the induced sums s_f(υi) = $f(v_0)$ must be distinct from $s_f(\upsilon i) = \sum_{i=0}^{n} f(v_i)$. This requires all n leaves to carry distinct fuzzy values, plus one different value for the center. Thus, n+1 fuzzy labels are necessary and sufficient. Therefore, $\eta_f\ *$ ($K_{1,n}$) = n + 1.

IV. Bull Graph

Definition IV.1. The bull graph BG contains 5 vertices

$V(BG) = \{u_1,u_2,u_3,u_4,v_1\}$

and 5 edges

$E(BG) = \{u_iu_{i+1} : 1 \leq i \leq 3\} \cup \{ v_1\ u_i : 2 \leq i \leq 3\}$.

Refer Figure 3.

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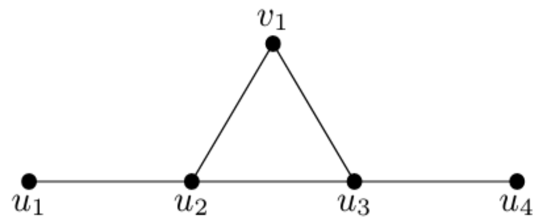

Fig. 3: Bull graph

Theorem IV.1. The bull graph is a fuzzy lucky graph with fuzzy lucky number η(BG) = 2.

Proof. Let V(BG) = { u_1,u_2,u_3,u_4,v_1} be the vertex set of the bull graph with edge set

E(BG) ={u_iu_{i+1} : $1 \leq i \leq 3$}∪{υ_1u_i : $2 \leq i \leq 3$}.

Define a fuzzy labeling function

$$f : V(BG) \to \{\mu_1,\mu_2\} \subset (0,1]$$

By

$$f\,(u) = \begin{cases}\mu_1, & i \neq 4\\ \mu_2, & i \neq 4\end{cases} \qquad f(v_1) = \mu_1 .$$

Now compute the induced sums:

$s_f\,(u_1) = f(v_1) = \mu_1$,

$s_f\,(u_2) = f(u_1) + f(u_3) + f(v_1) = 3\mu_1$,

$s_f\,(u_3) = f(u_2) + f(u_4) + f(v_1) = 2\,\mu_1 + \mu_2$,

$s_f\,(u_4) = f(u_3) = \mu_1$,

$$s_f\,(v_1) = f(u_2) + f(u_3) = 2\mu_1 .$$

Clearly, for every edge uυ ∈ E(BG), we have $s_f\,(u) \neq s_f\,(v)$ (since μ1≠ μ2). Thus, f is a fuzzy lucky labeling of the bull graph.

Therefore, the bull graph BG is a fuzzy lucky graph with fuzzy lucky number η_f (BG) = 2.

Example IV.1. A fuzzy lucky labelled bull graph is shown in Figure 4.

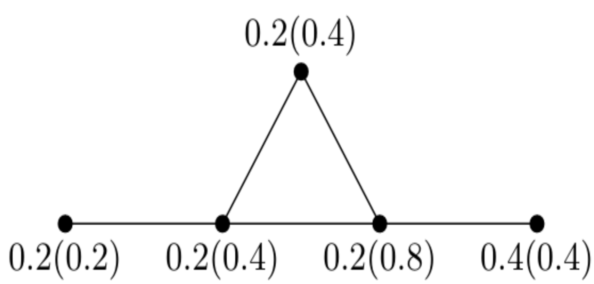

Fig. 4: FL Labeled Bull graph

Corollary IV.1.1. The fuzzy lucky number of bull graph is one more than its minimum degree, i.e.,

$$\eta_f(BG) = \delta(BG)+1.$$

Corollary IV.1.2. The fuzzy lucky number of bull graph is one less than its maximum degree, i.e.,

$$\eta_f\,(BG) = \Delta(BG)-1.$$

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Theorem IV.2. The bull graph is a proper fuzzy lucky graph with proper fuzzy lucky number η_f*(BG) = 3.

Proof. Let V(BG) = { u_1,u_2,u_3,u_4,v_1} and

E(BG) ={u_iu_{i+1} : $1 \leq i \leq 3$}∪{υ_1u_i : $2 \leq i \leq 3$}.

Define a fuzzy labeling function

f : V(BG) →{μ1,μ2,μ3} ⊂ (0,1]

by

$$f(u_i) = \begin{cases} \mu_1, & i = 2,4, \\ \mu_2, & i = 1,3 \end{cases} \qquad f(v_1) = \mu_3$$

where μ_1,μ_2,μ_3 are distinct fuzzy numbers in (0,1].

Now computing the induced sums:

$s_f(u_1) = f(u_2) = \mu_1$,

$s_f(u_2) = f(u_1) + f(u_3) + f(v_1) = 2\mu_1 + \mu_3$,

$s_f(u_3) = f(u_2) + f(u_4) + f(v_1) = 2\mu_1 + \mu_3$,

$s_f(u_4) = f(u_3) = \mu_2$,

$s_f(v_1) = f(u_2) + f(u_3) = \mu_1 + \mu_2$.

Clearly, for every edge uv ∈ E(BG), we have $s_f(u) \neq s_f(v)$ and $f(u) \neq f(v)$. Thus f is a Proper Fuzzy Lucky Labeling of BG. Therefore, the bull graph BG is a proper fuzzy lucky graph with proper fuzzy lucky number η_f*(BG) = 3.

Example IV.2. A proper fuzzy lucky labelled bull graph is shown in Figure 5.

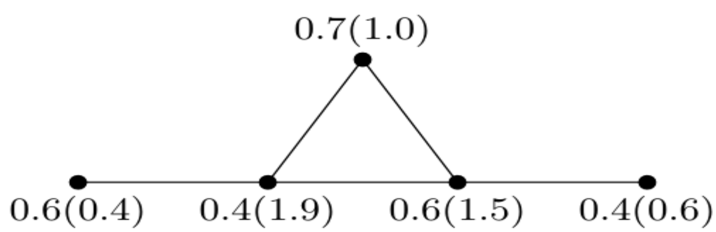

Fig. 5: PFL Labeled Bull graph

V. Fork Graph

Definition V.1. The fork graph FKG contains 5 vertices

V(FKG) ={$u_1,u_2,u_3,\upsilon_1,\upsilon_2$}

and 4 edges

E(FKG) ={u_iu_{i+1} : $1 \leq i \leq 2$}∪{ υ_1u_3, υ_2u_2}.

Refer Figure 6.

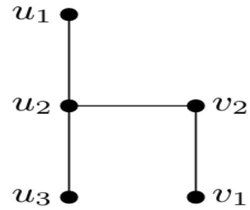

Fig. 6: Fork graph

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Corollary V.1. If a graph G is locally irregular (i.e., deg(u) ≠deg(υ) for every edge uυ ∈ E(G)), then η_f(G) = 1.

Proof. Label every vertex with the same fuzzy value μ_1 >0. Then $s_f\,(u) = \deg(u)\mu_1$ and $s_f\,(\upsilon) = \deg(\upsilon)\mu_1$ differ on each edge since the incident degrees differ. Hence this constant labeling is fuzzy lucky, using one fuzzy value.

Remark. The fork graph FKG is locally irregular, so the corollary yields η_f(FKG) = 1, agreeing with the theorem above.

Theorem V.1. The fork graph FKG is a fuzzy lucky graph with fuzzy lucky number η_f(FKG) = 1. Proof. Let V (FKG) = {u_1,u_2,u_3, υ_1, υ_2} and E(FKG) ={u_1u_2, u_2u_3, $u_3\,\upsilon_1$, $u_2\,\upsilon_2$}.

Define a constant fuzzy labelling

f: V(FKG) →{μ_1}⊂(0,1],

f(x) = μ_1 for all x ∈ V(FKG).

Then for each vertex,

$$s_f(x) = \sum_{xy \epsilon E(FKG)} f(y) = (x)\mu_1\,.$$

Degrees in FKG are

deg(u_1) = 1, deg(u_2) = 3,

deg(u_3) = 2, deg(υ_1) = 1,

deg(υ_2) = 1.

Hence, along every edge the induced sums differ:

$s_f(u_1)$ = μ_1 ≠ 3μ_1 = $s_f(u_2)$,

$s_f(u_2)$ = 3μ_1 ≠ 2μ_1 = $s_f(u_3)$,

$s_f(u_3)$= 2μ_1 ≠ μ_1 = $s_f(\upsilon_1)$,

$s_f(u_2)$ = 3μ_1 ≠ μ_1 = $s_f(\upsilon_2)$.

Therefore f is a fuzzy lucky labeling. Since a single fuzzy value suffices,

η_f(FKG) = 1.

Example V.1. A fuzzy lucky labelled fork graph is shown in Figure 7.

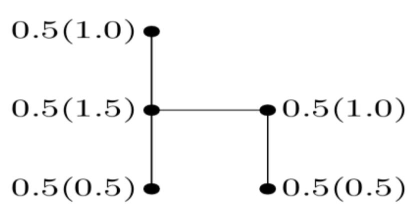

Fig. 7: FL Labeled Fork graph

Corollary V.1.1. If a graph G is locally irregular (i.e., deg(u) ≠ deg(υ) for every edge uv ∈ E(G)), then η_f(G) = 1.

Proof. Label every vertex with the same fuzzy value μ_1 >0. Then $s_f(u)$ = deg(υ)μ_1 and $s_f(\upsilon)$ = deg(υ)μ_1 differ on each edge since the incident degrees differ. Hence this constant labeling is fuzzy lucky, using one fuzzy value. Remark. The fork graph FKG is locally irregular, so the corollary yields η_f(FKG) = 1, agreeing with the theorem above.

Theorem V.2. The fork graph FKG is a proper fuzzy lucky graph with a proper fuzzy lucky number

Therefore, f is a fuzzy lucky labeling. Since a single fuzzy value suffices,

η_f ∗(FKG) = 2.

Proof. Let V (FKG) = {u_1,u_2,u_3, υ_1,υ_2} and

E(FKG) ={u_1u_2, u_2u_3, $u_3\,\upsilon_1$, $u_2\,\upsilon_2$}.

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Choose two distinct fuzzy numbers $\mu_1,\mu_2 \in (0,1]$

Define

$$(u_i) = \begin{cases} \mu_1, & if \ = 2, \\ \mu_2, & i \neq 2. \end{cases}$$

$$f\,(\upsilon_i) = \begin{cases} \mu_1, & i = 1, \\ \mu_2, & i = 2. \end{cases}$$

Then adjacent vertices receive distinct labels, so f is proper. The induced sums are

$s_f\,(u_1) = f(u_2) = \mu_1$,

$s_f\,(u_2) = f(u_1) + f(u_3) + f(v_2) = 3\mu_2$,

$s_f\,(u_3) = f(u_2) + f(v_1) = 2\,\mu_1$,

$s_f\,(v_1) = f(u_3) = \mu_2$,

$$s_f\,(v_2) = f(u_2) = \mu_1.$$

Along every edge the sums differ:

$\mu_1 \neq 3\mu_2$, $3\mu_2 \neq 2\mu_1$,

$2\mu_1 \neq \mu_2$, $3\mu_2 \neq \mu_1$.

Hence f is a Proper Fuzzy Lucky Labeling of FKG, showing $\eta_f\,*$(FKG) ≤ 2. Since a Proper Fuzzy Lucky Labeling is a proper coloring, we must use at least χ(FKG) = 2 labels; therefore $\eta_f\,*$(FKG) ≥2. Combining, $\eta_f\,*$(FKG) = 2.

Example V.2. A proper fuzzy lucky labelled fork graph is shown in Figure 8.

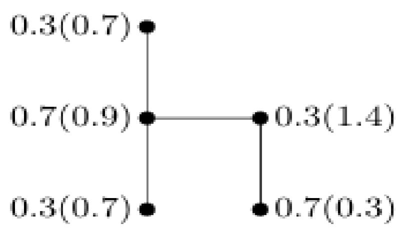

Fig. 8: PFL Labeled Fork graph

Corollary V.2.1. For the fork graph, $\eta * f$(FKG) = δ(FKG)+1.

Proof. The degree sequence of FKG is (1,3,2,1,1), so δ(FKG) = 1. By the theorem, $\eta_f *$(FKG) = 2 = δ(FKG)+1.

VI. Diamond Graph

Definition VI.1. The diamond graph DG contains 4 vertices

$$V(DG) = \{u_1,u_2,u_3,v_1\}$$

and 5 edges

$$E(DG) = \{u_iu_{i+1} : 1 \leq i \leq 2\} \cup \{\upsilon_1 u_i : 1 \leq i \leq 3\}.$$

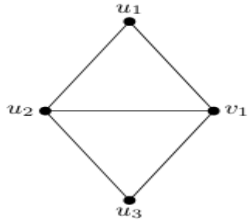

Fig. 9: Diamond graph

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Refer Figure 9.

Theorem VI.1. The diamond graph DG is a fuzzy lucky graph with fuzzy lucky number $\eta_f(DG) = 2$.

Proof. Let V (DG) = {u_1,u_2,u_3,υ_1} and E(DG) ={u_1u_2, u_2u_3, υ_1u_1, υ_1u_2, υ_1u_3}. Choose two distinct fuzzy numbers $\mu_1,\mu_2 \in (0,1]$ and define

$f(u_1) = \mu_1$, $f(u_2) = \mu_2$,

$f(u_3) = \mu_2$, $f(\upsilon_1) = \mu_1$.

The induced sums are

$s_f\,(u_1) = f(u_2) + f(v_2) = \mu_1 + \mu_2$,

$s_f\,(u_2) = f(u_1) + f(u_3) + f(v_1) = 2\mu_1 + \mu_2$,

$s_f\,(u_3) = f(u_2) + f(v_1) = \mu_2 + \mu_1$,

$s_f\,(v_2) = f(u_1) + f(u_2) + f(u_3) = \mu_1 + 2\mu_2$.

Along each edge the sums differ:

u_1u_2 : $\mu_1 + \mu_2 \neq 2\mu_1 + \mu_2$,

u_2u_3 : $2\mu_1 + \mu_2 \neq \mu_1 + \mu_2$,

υ_1u_1 : $\mu_1 + 2\mu_2 \neq \mu_1 + \mu_2$,

υ_1u_2 : $\mu_1 + 2\mu_2 \neq 2\mu_1 + \mu_2$ (since $\mu_1 \neq \mu_2$),

υ_1u_3 : $\mu_1 + 2\mu_2 \neq \mu_1 + \mu 2$.

Hence f is a fuzzy lucky labeling, so $\eta_f(DG) \leq 2$. A single-label assignment $f \equiv \mu$ fails because u_2 and υ_1 are adjacent and both have degree 3, which would give $s_f(u_2) = s_f(\upsilon_1) = 3\mu$. Thus at least two labels are required, and η_f (DG) = 2.

Example VI.1. A fuzzy lucky labelled diamond graph is shown in Figure 9.

Corollary VI.1.1. For the diamond graph, $\eta_f(DG) = \Delta(DG)-1=2$. Proof. Here $\Delta(DG) = 3$ (vertices u_2 and υ_1). By the theorem, $\eta_f(DG) = 2$. Since two adjacent vertices

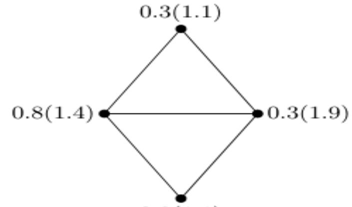

Fig. 10: FL Labeled Diamond graph

have the same maximum degree 3, any single-label fuzzy assignment would yield equal induced sums on that edge, so one label is impossible. Therefore $\eta_f(DG) = 2 = \Delta(DG)-1$.

Theorem VI.2. The diamond graph DG is a proper fuzzy lucky graph with proper fuzzy lucky $_2$number $\eta_f*(DG) = 3$.

Proof. Let V (DG) = {u_1,u_2,u_3,υ_1} and

E(DG) ={u_1u_2, u_2u_3, υ_1u_1, υ_1u_2, υ_1u_3}.

Choose three distinct fuzzy numbers $\mu_1,\mu_2,\mu_3 \in (0,1]$ such that Define

$f(u_1) = \mu_1$, $f(u_2) = \mu_2$,

$f(u_3) = \mu_1$, $f(\upsilon_1) = \mu_3$.

The induced sums are

$s_f\,(u_1) = f(u_2) + f(v_1) = \mu_2 + \mu_3$,

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$s_f(u_2) = f(u_1) + f(u_3) + f(v_1) = 2\mu_1 + \mu_3$,

$s_f(u_3) = f(u_2) + f(v_1) = \mu_2 + \mu_3$,

$s_f(v_1) = f(u_1) + f(u_2) + f(u_3) = 2\mu_1 + \mu_2$.

Along each edge the sums differ:

u_1u_2 : $\mu_2 + \mu_3 \neq 2\mu_1 + \mu_3$ ($\mu_2 \neq 2\mu_1$),

u_2u_3 : $2\mu_1 + \mu_3 \neq \mu_2 + \mu_3$ ($2\mu_1 \neq \mu_2$),

v_1u_1 : $2\mu_1 + \mu_2 \neq \mu_2 + \mu3$ ($2\mu_1 \neq \mu_3$),

v_1u_2 : $2\mu_1 + \mu_2 \neq 2\mu_1 + \mu3$ ($\mu_2 \neq \mu_3$),

v_1u_3 : $2\mu_1 + \mu_2 \neq \mu_2 + \mu_3$ ($2\mu_1 \neq \mu_3$).

Hence f is a Proper Fuzzy Lucky Labeling, so $\eta_f*(DG) \leq 3$. Since DG contains the triangle $v_1u_1u_2\,v_1$, we have $\chi(DG) = 3$, and any Proper Fuzzy Lucky Labeling uses at least $\chi(DG)$ labels; thus $\eta_f*(DG) \geq 3$. Therefore $\eta_f*(DG) = 3$.

Example VI.2. A proper fuzzy lucky labelled diamond graph is shown in Figure 11.

Corollary VI.2.1. For the diamond graph,

$$\eta_f*(DG) = \chi(DG) = \Delta(DG) = 3$$

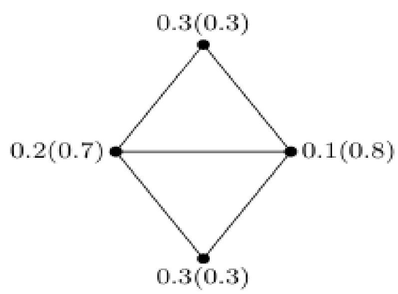

Fig. 11: PFL Labeled Diamond graph

Proof. The graph DG has a triangle, so $\chi(DG) = 3$; its maximum degree is $\Delta(DG) = 3$ (at u_2 and υ_1); and the theorem gives $\eta_f*(DG) = 3$, hence all three values coincide.

VII. Conclusion

The concept of fuzzy lucky labeling and proper fuzzy lucky labeling provides a natural extension of classical labeling theory into fuzzy environments. The study demonstrates that all connected graphs admit fuzzy lucky labelings, while trees and several classes of special graphs admit proper fuzzy lucky labelings. The relationship between the proper fuzzy lucky number and graph parameters such as the chromatic number and degree properties was also established.

For specific cases such as star graphs, bull graphs, and fork graphs, exact fuzzy lucky numbers and proper fuzzy lucky numbers were determined, highlighting the practical computability of these parameters. This research contributes to the growing intersection of fuzzy mathematics and graph theory, offering potential applications in uncertain or imprecise systems such as communication networks, resource allocation, and computational intelligence.

References

1. A. Rosa, On certain valuations of the vertices of a graph, in: Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, New York and Dunod, Paris, 1967, pp. 349–355.
2. J. Gallian, A dynamic survey of graph labeling, Electronic Journal of Combinatorics, DS6 (first edition 1997, regularly updated).
3. F. Harary, Sum graphs and difference graphs, Congressus Numerantium, 72 (1990), 101–108.
4. G. Chen, Integral sum graphs, Discrete Mathematics, 124 (1994), 67–74.
5. M. Karo´nski, T. Łuczak, A. Thomason, Edge weights and vertex colours, Journal of Combinatorial Theory, Series B, 91 (2004), 151–157.
6. A. Dehghan, A. Sadeghi, B. Ahadi, Algorithmic complexity of proper labeling problems, Theoretical Computer Science, 495 (2013), 25–36.

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7. J. Czerwi´nski, J. Grytczuk, W. Łazny, Lucky labelings of graphs, Information Processing Letters, 109 (2009), 1078–1081.
8. B. Ahadi, A. Dehghan, M. Kazemi, Determining the lucky number of a graph is NP-hard, Discrete Applied Mathematics, 160 (2012), 2420–2426.
9. S. Akbari, M. Ghanbari, A. Jahanbekam, On lucky choice number of graphs, Graphs and Combinatorics, 29 (2013), 1179–1188.
10. C. Kujur, C. Lal, Lucky labeling of bloom graphs, Journal of Discrete Mathematical Sciences and Cryptography, 20 (2017), 1239–1246.
11. V. Antony Xavier, et al., Proper lucky labeling of hexagonal, triangular and Sierpi´nski graphs, International Journal of Pure and Applied Mathematics, 113 (2017), 325–336.
12. R. Vidya, Product cordial labeling of alternate triangular and quadrilateral snake graphs, International Journal of Computer Applications, 88 (2014), 1–5.
13. P. Lavanya, S. Sankari, Graceful labeling of quadrilateral snake graphs, International Journal of Mathematical Archive, 8 (2017), 198–202.
14. N. Ramya, Lucky edge labeling of split star and snake graphs, International Journal of Pure and Applied Mathematics, 115 (2017), 489–497.
15. P. Prajapati, SD-prime cordial labeling for snake graphs, International Journal of Mathematics Trends and Technology, 67 (2019), 58–65.
16. A. Sunoj, Square difference prime labeling of snake graphs, International Journal of Scientific Research in Mathematical and Statistical Sciences, 4 (2017), 13–20.
17. M. Jadav, H. Ghodasara, Strongly* labeling in snake-related graphs, International Journal of Mathematics Trends and Tech nology, 25 (2015), 43–49.
18. N. A. Sudibyo, Total vertex irregularity strength of ladder-type graphs, Journal of Physics: Conference Series, 1211 (2019), 012010.
19. N. Rathi, Quotient labeling of ladder and related graphs, International Journal of Mathematics and Its Applications, 6 (2018), 195–207.
20. S. Meenakshi, Fuzzy lucky labeling of quadrilateral graphs, International Journal of Engineering and Technology, 10 (2021), 245–252.
21. P. Kumar, S. Meenakshi, Fuzzy lucky labeling of jewel, comb and fan graphs, International Journal of Computer Applications, 175 (2020), 25–31.
22. C. Karthikeyan, et al., Fuzzy lucky labeling of double star graphs, Journal of Mathematical and Computational Science, 15 (2025), 112–125.
23. M. Badr, Odd-graceful labeling of triangular snake graphs, International Journal of Pure and Applied Mathematics, 85 (2013), 315–324.
24. A. Moussa, Labeling in ladder graph subdivisions, International Mathematical Forum, 11 (2016), 123–132.
25. K. Aishwarya, Lucky edge labeling of shell and book graphs, International Journal of Advanced Research in Mathematics, 3 (2015), 65–72.
26. R. Sandhya, Root square mean labeling of bull and cricket graphs, International Journal of Mathematics Trends and Technology, 66 (2020), 102–110.
27. B. Annamma, K. Begum, Prime labeling of star-related mirror graphs, International Journal of Pure and Applied Mathematics, 125 (2020), 223–231.
28. S. Senthurpriya, S. Meenakshi, Labeling methods in domination problems, Journal of Computer and Mathematical Sciences, 10 (2019), 512–520.
29. M. Sudhakar, et al., Cordial labeling of star graphs, International Journal of Computer Applications, 180 (2018), 33–40.
30. N. Murugan, K. Chitra, Lucky edge labeling of triangular graphs, International Journal of Mathematics and Soft Computing, 6 (2016), 97–106.
31. M. Thirusangu, Proper d-lucky labeling of quadrilateral snake graphs, International Journal of Pure and Applied Mathematics, 119 (2019), 247–256.
32. C. Chiranjilal Kujur, Proper d-lucky labeling of corona and rooted product graphs, Journal of Discrete Mathematical Sciences and Cryptography, 22 (2019), 1245–1260.