b - Generalized Pre-Closed Sets in Topological Spaces

M. Andal[1], V. Thiripurasundari[2]

[1]Assistant Professor in Department of Mathematics, P.S.R.R College of Engineering, Sivakasi, Affiliated to Anna University, Chennai

[2]Associate Professor in Department of Mathematics, Sri S. Ramasamy Naidu Memorial College, Sattur, Affiliated to Maduari Kamaraj University, Madurai

Abstract: In this paper we introduce a new class of b-generalized closed sets, b- generalized open sets in topological spaces, and study some of its basic properties.

I. Introduction

The study of generalized closed sets in topological spaces was initiated by Levine[3] in 1970. Andrijevic[1] introduced a new class of generalized open sets in a topological space, the so called b-open sets. Maki et.al[2] defined αg – closed set and gα – closed set in 1994. Balachandran and Rani[5] defined the notion of generalized pre-closed sets. A.A.Omari[4] and M.S.M.Noorani introduced and studied the concept of generalized b- closed sets in topological spaces.

Preliminaries

Definition 2.1

A topology on a set X is a collection τ of subsets of X having the following properties:

i. $\emptyset$ and X are in τ.

ii. The union of the elements of any subcollection of τ is in τ.

iii. The intersection of the elements of any finite subcollection of τ is in τ. A set X for which a topology τ has been specified is called a topological space.

Definition 2.2

Let (X, τ) be any topological spaces. A subset S of X is said to be a closed set in (X, τ) if its complement in X, namely X\S, is open in (X, τ).

Definition 2.3[5]

A subset A of a topological space X is called a pre - open if $A \subseteq int(cl(A))$ and pre - closed if $cl(int(A)) \subseteq A$.

Definition 2.4[5]

The pre closure of a subset A of X, denoted by pcl(A) is the intersection of all pre closed sets containing A. The pre interior of a subset A of X, denoted by pint(A) is the union of all pre open sets contained in A.

Definition 2.5[6]

A subset A of a topological space X is called a α - open if $A \subseteq int(cl(int(A)))$ and α - closed if $cl(int(cl(A))) \subseteq A$.

Definition 2.6[1]

A subset A of a topological space X is called a b - open if $A \subseteq cl(int(A)) \cup int(cl(A))$ and b - closed if $cl(int(A)) \cap int(cl(A)) \subseteq A$.

Definition 2.7[7]

A subset A of a topological space X is called a regular - open if $A = int(cl(A))$ and regular - closed if $cl(int(A)) = A$.

Definition 2.8[3]

A subset A of a topological space X is called a generalized closed (briefly, g - closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.9[6]

A subset A of a topological space X is called a generalized α - closed (briefly, gα - closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α - open in X.

Definition 2.10[6]

A subset A of a topological space X is called a α - generalized closed (briefly, αg - closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.11[2]

A subset A of a topological space X is called a generalized pre - closed (briefly, gp - closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.12[4]

A subset A of a topological space X is called a generalized b - closed (briefly, gb - closed) if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

3. b - Generalized Pre Closed sets.

Definition 3.1

A subset A of a topological space X is called a b - generalized pre closed (briefly, bgp - closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is b - open in X.

The family of all bgp - closed sets in a topological space X is denoted by bgpc(X).

Example 3.2

Let X = {a, b, c} and topology τ = {X, ∅, {b}, {b, c}}, then bgpc(X) = {X, ∅, {a}, {c}, {a, c}}.

Theorem 3.3

Every closed set is a bgp - closed set.

Proof: Let A be any closed set in X such that $A \subseteq U$ where U is b - open set in X. Since A is closed set and $pcl(A) \subseteq cl(A)$, $pcl(A) \subseteq U$ and U is b - open in X. Hence A is a bgp - closed set.

Remark 3.4

The converse of the theorem need not be true as seen from the following example.

Example 3.5

Consider X = {a, b, c} with the topology τ = {X, ∅, {b}, {b, c}}. Let A = {c}, then A is bgp - closed set but not a closed set.

Theorem 3.6

Every pre - closed set is a bgp - closed set.

Proof: Let A be a pre - closed set in X such that $A \subseteq U$ where U is b - open in X. Since A is pre - closed and $pcl(A) = A$, $pcl(A) \subseteq U$ and U is b - open in X. Hence A is a bgp - closed set.

Remark 3.7

The converse of the theorem need not be true as seen from the following example.

Example 3.8

Consider X = {a, b, c} with the topology τ = {X, ∅, {a}, {a, b}}. Let A = {a, c}, then A is bgp - closed set but not a pre closed set.

Theorem 3.9

Every regular - closed set is a bgp - closed set.

Proof: Let A be a regular closed set in X such that $A \subseteq U$ where U is b - open in X. Since A is regular closed set and $pcl(A) \subseteq rcl(A)$, $pcl(A) \subseteq U$. Hence A is a bgp - closed set.

Remark 3.10

The converse of the theorem need not be true as seen from the following example.

Example 3.11

Consider X = {a, b, c} with the topology τ = {X, ∅, {b}, {b, c}}. Let A = {a}, then A is bgp - closed set but not a regular closed set.

Theorem 3.12

Every α - closed set is a bgp - closed set.

Proof: Let A be a α - closed set in X such that A ⊆ U where U is b - open in X. Since A is a α - closed set and pcl(A) ⊆ αcl(A), pcl(A) ⊆ U. Hence A is a bgp - closed set.

Remark 3.13

The converse of the theorem need not be true as seen from the following example.

Example 3.14

Consider X = {a, b, c} with the topology τ = {X, ∅, {a}, {a, b}}. Let A = {a, c}, then A is bgp - closed set but not a α - closed set.

Theorem 3.15

Every gα - closed set is a bgp - closed set.

Proof: Let A be a gα - closed set in X such that A ⊆ U where U is b - open in X. Since A is a gα - closed set and pcl(A) ⊆ αcl(A), pcl(A) ⊆ U. Hence A is a bgp - closed set.

Remark 3.16

The converse of the theorem need not be true as seen from the following example.

Example 3.17

Consider X = {a, b, c} with the topology τ = {X, ∅, {a}, {a, b}}. Let A = {a, c}, then A is bgp - closed set but not a gα - closed set.

Theorem 3.18

Every gp - closed set is a bgp - closed set.

Proof: Let A be a gp - closed set in X such that A ⊆ U where U is b - open in X. Since A is a gp - closed set and pcl(A) ⊆ U. Hence A is a bgp - closed set.

Remark 3.19

The converse of the theorem need not be true as seen from the following example.

Example 3.20

Consider X = {a, b, c} with the topology τ = {X, ∅, {a, b}}. Let A = {a}, then A is bgp - closed set but not a gp - closed set.

Theorem 3.21

Every g - closed set is a bgp - closed set.

Proof: Let A be a g - closed set in X such that A ⊆ U where U is b - open in X. Since A is a g - closed set and pcl(A) ⊆ cl(A), pcl(A) ⊆ U. Hence A is a bgp - closed set.

Remark 3.22

The reverse implication of the above theorem need not be true as seen from the following example.

Example 3.23

Consider X = {a, b, c} with the topology τ = {X, ∅, {b}, {b, c}}. Let A = {c}, then A is bgp - closed set but not a g - closed set.

Theorem 3.24

Every αg - closed set is a bgp - closed set.

Proof: Let A be a g - closed set in X such that A ⊆ U where U is b - open in X. Since A is a αg - closed set and pcl(A) ⊆ αcl(A), pcl(A) ⊆ U. Hence A is a bgp - closed set.
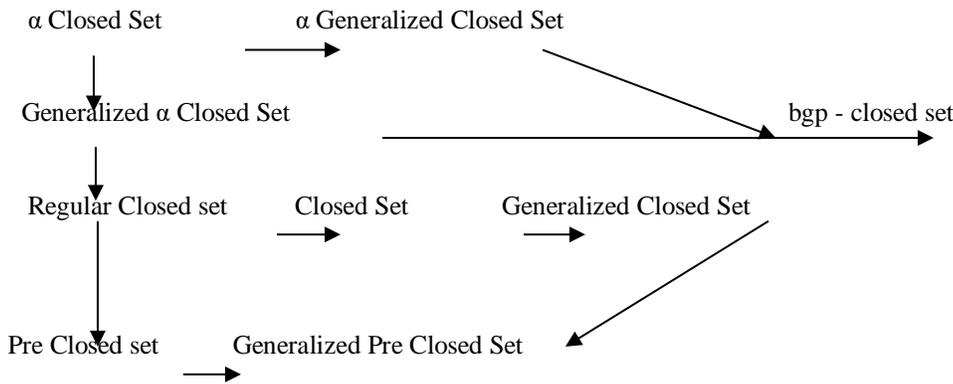
Remark 3.25

The converse of the above theorem need not be true as seen from the following example.

Example 3.26

Consider X = {a, b, c} with the topology τ = {X, ∅, {a, b}}. Let A = {b}, then A is bgp - closed set but not a αg - closed set.

Remark 3.27

The Reverse implication of the above theorem need not be true summarized in the following implicit diagram.

Diagram 1

Theorem 3.28

Let $B \subseteq X$ be a bgp - closed set, then pcl(B) - B contains no non - empty b - closed set.

Proof: Let B be a bgp - closed set and M be a b - closed set in X such that $M \subseteq$ pcl(B) - B, then $M \subseteq$ pcl(B) and $M \subseteq X$ - B implies $B \subseteq X$ - M. Now, B is a bgp - closed set and X – M is a b - open set containing B. It follows that pcl(B) $\subseteq X$ - M and thus $M \subseteq X$ - pcl(B). This implies $M \subseteq$ pcl(B) $\cap$ (X - pcl(B)) $= \emptyset$. Hence $M = \emptyset$. Therefore pcl(B) - B contains no non - empty b - closed set.

Theorem 3.29

Let $A \subseteq X$ be a bgp - closed set, then A is pre - closed if and only if pcl(A) - A is b – closed set.

Proof: Let $A \subseteq X$ be a bgp - closed set. Let A be a pre - closed set in X, then we have pcl(A) - A = $\emptyset$ which is b - closed set. Therefore pcl(A) - A is b - closed set. Conversely, Assume that pcl(A) - A is b - closed set. Now, A is bgp - closed set. Since pcl(A) - A is a b - closed subset itself. By theorem 3.27, pcl(A) - A = $\emptyset$. This implies that pcl(A) = A and so A is pre - closed. Therefore A is pre - closed.

Theorem 3.30

If $A \subseteq X$ is both b - open and bgp - closed, then A is pre - closed in X.

Proof: Let A be b - open and bgp - closed set in X, then pcl(A) $\subseteq$ A. But $A \subseteq$ pcl(A) is always true. Therefore pcl(A) = A. Hence A is a pre - closed set.

Theorem 3.31

If $A \subseteq X$ is a bgp - closed set and $A \subseteq B \subseteq$ pcl(A), then B is bgp – closed in X.

Proof: Let U be a b - open set in X such that $B \subseteq U$, then $A \subseteq U$. Since A is a bgp - closed, then pcl(A) $\subseteq U$ and $A \subseteq B \subseteq$ pcl(A). Now, pcl(B) $\subseteq$ pcl(pcl(A)) = pcl(A) $\subseteq U$. So, pcl(B) $\subseteq U$.

Remark 3.32

The converse of the above theorem need not be true as seen from the following example.

Example 3.33

Consider X ={a, b, c} with the topology τ = {$\emptyset$, X, {a}}. Let A = {a}, B = {a, b}, then $A \subseteq B \subseteq$ pcl(A) and B is bgp - closed set in X but A is not a bgp - closed set in X.

Theorem 3.34

Let Y be an open subspace of a space X, and $A \subseteq Y$. If A is bgp - closed set in X, then A is bgp - closed set in Y.

Proof: Let U be a b - open set of Y such that $A \subseteq U$. Then U = Y $\cap$ H for some b - open set H of X. Since A is bgp - closed set in X, we have pcl(A) $\subseteq U$ and $pcl_Y(A) = Y \cap pcl(A) \subseteq Y \cap H = U$. Hence A is a bgp closed set in Y.

Remark 3.35

The converse of the above theorem need not be true as seen from the following example.

Example 3.36

Consider X ={a, b, c} with the topology τ = {∅, X, {a, b}} and let Y = {a, b} with the topology τ = {∅, X, {a, b}}. Let A = {a, b}, then A ⊆ Y ⊆ X and A is bgp - closed set relative to Y but it is not bgp - closed relative to X.

Theorem 3.37

For a space X, the following statements are equivalent:

(i) Every bgp closed set is gb - closed set.

(ii) Every pre closed set is gb - closed.

Proof: (i)→(ii):

Given, every bgp - closed set is gp - closed. By above theorem, "every pre - closed set is bgp - closed set". By our assumption, every pre closed set is gb - closed set.

(ii)→(i):

Given, every pre - closed set is gb - closed set. Let A be a bgp - closed set in X such that A ⊆ U where U is b - open in X, then pcl(A) ⊆ U. Since pcl(A) is pre - closed set, then by (ii), pcl(A) is gb - closed set. Therefore cl(A) ⊆ cl(pcl(A)) ⊆ U. That is, cl(A) ⊆ U. Therefore A is a gb - closed set. Hence every bgp closed set is gb - closed set.

4. b - Generalized Pre Open Sets

Definition 4.1

A subset A of a topological space X is called a b - generalized pre open (briefly, bgp - open) set if A^C is bgp - closed.

Theorem 4.2

A set A ⊆ X is bgp - open if and only if U ⊆ pint(A) whenever U is b - closed and U ⊆ A.

Proof: Let A be a bgp - open set. Suppose that U ⊆ A where U is b - closed set. Then X - A is a bgp - closed set contained in the b-open set X - U, pcl(X - A) ⊆ X - U. Since pcl(X - A) = X - pint(A), then X - pint(A) ⊆ X – U. That is U ⊆ pint(A). Conversely, let U ⊆ pint(A) whenever U ⊆ A and U is b - closed set, then X - pint(A) ⊆ X – U. That is pcl(X - A) ⊆ X – U. This implies X-A is bgp - closed set and A is bgp - open in X.

Theorem 4.3

If A is bgp - open set and B is any set in X such that pint(A) ⊆ B ⊆ A, then B is bgp - open set in X.

Proof: Let A is bgp - open and B is any set in X such that pint(A) ⊆ B ⊆ A. It follows from the definition 4.1 and theorem 3.22. Hence B is bgp - open set in X.

Theorem 4.4

If a set A ⊆ X is bgp - closed set, then pcl(A)-A is bgp-open set in X.

Proof: Suppose that A is bgp-closed set and M is a b-closed set such that M ⊆ pcl(A) - A. By theorem 3.19, M =∅. Hence M ⊆ pint(pcl(A) - A). Therefore by theorem 4.2, pcl(A) - A is bgp - open set.

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