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Mathematical Analysis of a Prey Predator and Ammensal Model with Time Delay

Paparao. A. V

Department of Mathematics, JNTU-GV College of Engineering, Vizianagaram(A)Viziangarm-5325003 A. P, India

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ABSTRACT:

We study a three-species ecological model in which the first species (x) preys on the second species(y) and the second species exerts an amensal effect on the third species (z). Interactions between predator and prey include an explicit time delay(τ) and the system is formulated as a set of delay differential equations. We identify the positive coexistence equilibrium and perform a local stability analysis about this steady state. we derive sufficient conditions for a Hopf bifurcation driven by the delay parameter (τ). By treating (τ) as the bifurcation parameter, we determine critical delay values at which the coexistence equilibrium loses stability and periodic oscillations emerge. Numerical simulations implemented in MATLAB confirm the analytical predictions and illustrate the instability regimes and bifurcating limit cycles.

Keywords: Prey, Predator Co-existing state, local stability, Hopf bifurcation

AMS classification: 34 DXX

1. INTRODUCTION

Differential equations are most popular in explaining the mathematical models I ecology. The stability analysis concept is explained in detail by Braun [9] and Simon's [10]. The ecological models are initiated by studied Lokta [1] and Volterra [2]. The Mathematical models and its stability analysis discussed by Kapur [3, 4]. qualitative analysis plays a big role in analysing these models due to the difficulty in finding analytical solutions due to the non-linearity of the models arise ecology. The qualitative analyses of ecological models are widely studied by authors [5-7]. The stability of analysis of delay-differential equations are significant in ecology. The time delays are influence the dynamics of the system and tend to destabilize or stabilizes the system. The systems with delay arguments and the qualitative analysis are widely discussed by the authors [11-13]. The nature of the delay argument cause unbounded growth and extinction of populations leads to instability tendency of models. The delay argument may classify in to continuous, discrete, distributed etc. The time lags can be discrete or continuous. These lags will change the stable equilibrium to unstable or vice-versa. The delay models in population dynamics are widely studied by paparao [14-21]. In this paper we take a logistic growth model of three species for investigation. In this model first species is preying on second and second is ammensal to third species. A discrete time lag is incorporated in the interaction of first and second species. The model is studied by a couple of delay-differential equations. The co-existing equilibrium point is identified and discussed the dynamics at this point. Numerical simulation is carried out carried out in support of stability analysis. It is shown that the system exhibits instability trendies leads to Hopf bifurcation.

1. Formation of Mathematical Model:

The proposed ecological system can be modelled into the following system of equations given by

$$\frac{dx}{dt} = a_1 x\left(1 - \frac{x}{k_1}\right) + a_{12}x(t-\tau)y(t-\tau)$$

$\frac{dy}{dt} = a_2 y\left(1 - \frac{y}{k_2}\right) - a_{21}x(t-\tau)y(t-\tau)$ (2.1)

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$$\frac{dz}{dt} = a_3 z\left(1 - \frac{z}{k_3}\right) - a_{32} yz$$

2.1 Nomenclature:

Parameter	Description
x,y,z	predator, prey and amensal population density
a_1, a_2, a_3	Natural growth rates of predator, prey and amensal
α_{12}	Interaction rate of first and second species (positive value)
α_{21}	Interaction rate of second and first species (negative value)
α_{32}	Interaction rate of third (ammensal) and second species (negative value)
k_1, k_2, k_3	Carrying capacities of first, second and third species populations respectively.

3.Equilibrium Points

Equating the system of equations (2.1) to zero, and derive the co-existing state is given by

$$E(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} \frac{k_1(a_1 a_2 + a_{12} a_2 k_2)}{(a_1 a_2 + a_{12}\alpha_{21} k_1 k_2)}, \\ \frac{k_2(a_1 a_2 - \alpha_{21} a_1 k_1)}{(a_1 a_2 + a_{12}\alpha_{21} k_1 k_2)}, \\ \frac{k_3\{(a_3 a_1 a_2 + a_{12}\alpha_{21} k_1 k_2) - k_2 \alpha_{32}(a_1 a_2 + a_{21} a_1 k_1)]}{a_3(a_1 a_2 + a_{12}\alpha_{21} k_1 k_2)} \end{pmatrix} \quad (3.1)$$

Co-existing state exist if (i) $a_2 a_2 > a_1 \alpha_{21} k_1$

(ii) $(a_3 a_1 a_2 + a_{12}\alpha_{21} k_1 k_2) > k_2 \alpha_{32}(a_1 a_2 + a_{21} a_1 k_1)$ (3.2)

are Satisfied

4. Local Stability Analysis at Co-existing State

Theorem4.1: The co-existing state is locally asymptotically stable

Proof: The variational matrix for the system (2.1) is

$$J = \begin{bmatrix} a_1 - \frac{2a_1\overline{x}}{k_1} + a_{12}\overline{y}e^{-\lambda\tau} & a_{12}\overline{x}e^{-\lambda\tau} & 0 \\ -a_{21}\overline{y}e^{-\lambda\tau} & a_2 - \frac{2a_2\overline{y}}{k_2} + a_{12}\overline{y}e^{-\lambda\tau} & 0 \\ 0 & -a_{32}\overline{z} & a_3 - \frac{2a_3\overline{z}}{k_3} - a_{32}\overline{y} \end{bmatrix} \quad (4.1.1)$$

Characteristic equation of the (4.1.1) is given by $\psi(\lambda,\tau) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + e^{-\lambda\tau}(q_1\lambda^2 + q_2\lambda + q_3) = 0$ (4.1.2)

Where

$$P_1 = \frac{2a_1\overline{x}}{k_1} + \frac{2a_2\overline{y}}{k_2} + \frac{2a_3\overline{z}}{k_3} - (a_1 + a_2 + a_3)$$

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$$P_2 = a_1a_2 + a_1a_3 + a_2a_3 + \frac{4a_1a_2\overline{x}\overline{y}}{k_1k_2} + \frac{4a_1a_3\overline{x}\overline{z}}{k_1k_3} + \frac{4a_2a_3\overline{y}\overline{z}}{k_2k_2} + \frac{2a_1a_{32}\overline{x}\overline{y}}{k_1} + \frac{2a_2a_{32}\overline{y}^2}{k_2}$$
$$-\frac{2a_1a_2\overline{y}}{k_2} - \frac{2a_1a_2\overline{x}}{k_1} - \frac{2a_1a_3\overline{z}}{k_3} - \frac{2a_2a_3\overline{z}}{k_3} - a_{32}a_1\overline{y} - a_{32}a_2\overline{y} - \frac{2a_1a_3\overline{x}}{k_1} - \frac{2a_3a_2\overline{y}}{k_2}$$

$$P_3 = \frac{2a_1a_2a_3\overline{y}}{k_2} + \frac{2a_1a_2a_3\overline{x}}{k_1} + \frac{2a_1a_2a_3\overline{z}}{k_3} + \frac{4a_1a_2a_{32}\overline{x}\overline{y}^2}{k_1k_2} + \frac{8a_1a_2a_3\overline{x}\overline{y}\overline{z}}{k_1k_2k_3} + a_1a_2a_{32}\overline{y}$$
$$-\frac{4a_1a_2a_3\overline{y}\overline{z}}{k_2} - \frac{4a_1a_2a_3\overline{z}\overline{x}}{k_3} - \frac{2a_1a_3a_{32}\overline{y}^2}{k_2} - \frac{2a_1a_2a_{32}\overline{x}\overline{y}}{k_1} - a_1a_2a_3 - \frac{4a_1a_2a_3\overline{x}\overline{y}}{k_1k_2}$$

$$q_1 = a_{21}\overline{x} - a_{12}\overline{y}$$

$$q_2 = a_1a_{12}\overline{x} + a_2a_{12}\overline{y} + a_{12}a_3\overline{y} + a_{21}a_{32}\overline{x}\overline{y} + \frac{2a_3a_{21}\overline{x}\overline{z}}{k_3} - \frac{2a_2a_{12}\overline{y}^2}{k_2} - a_{12}a_{32}\overline{y}^2 - a_3a_{21}\overline{x} - a_1a_{21}\overline{x}$$

$$q_3 = a_1a_{21}a_3\overline{x} + \frac{2a_1a_{21}a_{32}\overline{x}\overline{y}}{k_1} + \frac{2a_{12}a_3a_2\overline{y}^2}{k_2} + 2a_{12}a_{32}a_2\overline{y}^2 + \frac{4a_1a_{21}a_3\overline{x}\overline{z}}{k_1k_3} + \frac{2a_2a_{12}a_3\overline{y}\overline{z}}{k_3}$$
$$-\frac{2a_1a_3a_{21}\overline{x}\overline{z}}{k_3} - \frac{4a_2a_3a_{12}\overline{y}^2}{k_2} - \frac{2a_2a_{32}a_{12}\overline{y}^3}{k_2} - \frac{2a_{12}a_3\overline{y}\overline{z}}{k_3} - a_{21}a_{32}\overline{x}\overline{y} - \frac{2a_1a_{21}a_3\overline{x}}{k_1} - a_{12}a_2a_3\overline{y}$$

Which can be written as $\psi(\lambda,\tau) = P(\omega) + Q(\omega)e^{-\tau\omega}$

Case (i) For $\boldsymbol{\tau = 0}$

The characteristic equation obtained from (4.1.2) by putting $\tau = 0$ given by the following equation

$$\psi(\lambda,0) = -\left(\frac{a_3\overline{z}}{k_3} + \lambda\right)\left[\lambda^2 + \lambda\left(\frac{a_1\overline{x}}{k_1} + \frac{a_2\overline{y}}{k_2}\right) + \left(\frac{a_1a_2\overline{x}\overline{y}}{k_1k_2} + a_{12}a_{21}\overline{x}\overline{y}\right)\right] = 0$$

$$\left(\frac{a_3\overline{z}}{k_3} + \lambda\right) = 0 \; or \; \left[\lambda^2 + \lambda\left(\frac{a_1\overline{x}}{k_1} + \frac{a_2\overline{y}}{k_2}\right) + \left(\frac{a_1a_2\overline{x}\overline{y}}{k_1k_2} + a_{12}a_{21}\overline{x}\overline{y}\right)\right] = 0$$

$$\lambda = -\frac{a_3\overline{z}}{k_3}$$

$$and \; \left[\lambda^2 + \lambda\left(\frac{a_1\overline{x}}{k_1} + \frac{a_2\overline{y}}{k_2}\right) + \left(\frac{a_1a_2\overline{xy}}{k_1k_2} + a_{12}a_{21}\overline{xy}\right)\right] = 0 \qquad (4.1.3)$$

One of the roots is negative i.e., $-\frac{a_3\overline{z}}{k_3}$

From the equation (4.1.3) find the remaining two roots. if the two roots have negative real roots if the trace of the equation $\left(\frac{-b}{a}\right)$ is negative and the determinant $\left(\frac{c}{a}\right)$ is positive.

The trace and determinant from the equation (4.1.3) are given as follows

Here the trace is $= \frac{-b}{a} = \frac{-\left(a_1\overline{x}k_2 + a_2\overline{y}k_1\right)}{k_1k_2} < 0$

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Determinant= $\frac{c}{a}=\frac{\left(a_1a_2+a_{12}a_{21}k_1k_2\right)\bar{x}\bar{y}}{k_1k_2}>0$

Therefore, the system (2.1) is locally asymptotically stable at co-existing state.

Therefore, the co-existing state is locally asymptotically stable.

Case (ii) Let $\tau>0$: Suppose there is a positive τ_0 such that the equation (4.1.2) has pair of purely imaginary root, let the roots be $\pm i\omega,\omega>0,$ therefore $i\omega$ satisfies the equation (4.1.2)

$$(i\omega)^3+p_1(i\omega)^2+p_2(i\omega)+p_3+e^{-i\omega\tau}(q_1(i\omega)^2+q_2(i\omega)+q_3)=0$$

$$-\omega^2p_1+p_3-q_1\omega^2\cos\omega\tau+q_3\cos\omega\tau+q_2\omega\sin\omega\tau+$$
$$i[-w^3+wp_2+q_2\omega\cos\omega\tau+q_1\omega^2\sin\omega\tau-q_3\sin\omega\tau]=0$$

Separating real and imaginary parts, we get

$$(q_3-q_1\omega^2)\cos\omega\tau+q_2\omega\sin\omega\tau=\omega^2p_1-p_3 \qquad (4.1.4)$$

$$q_2\omega\cos\omega\tau-(q_3-q_1\omega^2)\sin\omega\tau=\omega^3-\omega p_2 \qquad (4.1.5)$$

On adding, the two equations after squaring, we get

From the above equations we get the following equation (by squaring and add the two results)

$$(q_3-q_1\omega)^2+(q_2\omega)^2=(\omega^2p_1-p_3)^2+(\omega^3-\omega p_2)^2$$

$$\omega^6+\omega^4(p_1{}^2-2p_2-q_1{}^2)+\omega^2(p_2^2-2p_1p_3-q_2^2+2q_1q_3)+q_3{}^2+p_3{}^2=0$$

Let $\psi(p)=p^3+p^2N_1+pN_2+N_3=0$ (4.1.6)

Where

$$N_1=p_1^2-2p_1-q_1{}^2$$
$$N_2=p_2^2-2p_1p_2-q_2^2+2q_1q_3$$
$$N_3=q_3^3+p_3^3$$
$$p=\omega^2$$

$$\therefore\psi(p)=o$$

If we assume that $N_1>0,N_2>0,N_3>0$ then equation (4.1.2) has no positive real roots. Therefore, the equation (4.2) admits negative real roots. Hence, we can derive the conditions for existences of stability at equilibrium point.

Theorem 4.2 The system (2.1) is locally asymptotically stable at co-existing state for all τ, if the following conditions hold.

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(i). $(p_1+q_1)>0$, $(p_2+q_2)>0$, $(p_3+q_3)>0$
(ii). $N_1>0, N_2>0, N_3>0$

Proof: Any one of N_1, N_2, N_3 is negative. Then equation (4.1.2) has a positive Root ω_0

Eliminating $\cos\omega\lambda$, from the equations (4.1.4) & (4.1.5), we have

$$\cos\omega\tau = \frac{\begin{vmatrix} \omega^2 p_1 - p_3 & q_2\omega \\ \omega^3 - \omega p_2 & -(q_3 - q_1\omega^2) \end{vmatrix}}{\begin{vmatrix} q_3 - q_1\omega^2 & q_2\omega \\ q_2\omega & -(q_3 - q_1\omega^2) \end{vmatrix}}$$

($\because$ by using Cramer's rule in determinants)

$$\cos\omega\tau = \frac{\omega^2 p_1 q_3 - p_3 q_3 - \omega^4 p_1 q_1 + \omega^2 q_1 p_3 + q_3 w^4 - p_2 q_3 \omega^2}{q_3^2 + q_1^2\omega^4 - 2q_1 q_3 w^2 + q_2^2\omega^2}$$

$$\tau_k = \frac{1}{\omega_0}\cos^{-1}\left[\frac{\omega_0^4(q_3 - p_1 q_1) + \omega_0^2(p_1 q_3 + q_1 p_3 - p_2 q_3) - p_3 q_3}{q_1^2\omega_0^4 + \omega_0^2(q_2^2 - 2q_1 q_3) + q_3^2}\right] + \frac{2k\pi}{\omega_0}$$

where $k = 0,1,2,3..............$

5. Hopf Bifurcation

Theorem 5.1: The sufficient condition for the system (4.1.1) admits bifurcation along the co-existing state E if $\tau > \tau_0$ and locally asymptotically stable If $0 < \tau < \tau_0$

Proof: Hopf bifurcation occurs when the real part of $\lambda(t)$ become positive when $\tau > \tau_0$ and the steady state become unstable moreover, when τ passes through the critical value τ_0 .

To check this result, differentiating the equation (4.1.2) With respect to τ, we get

$$3\lambda^2\frac{d\lambda}{d\tau} + 2p_1\lambda\frac{d\lambda}{d\tau} + p_2\frac{d\lambda}{d\tau} + e^{-\lambda\tau}(2q_1\lambda\frac{d\lambda}{d\tau} + q_2\frac{d\lambda}{d\tau}) + (q_1\lambda^2 + q_2\lambda + q_3)(-\lambda - \lambda\frac{d\lambda}{d\tau})e^{-\lambda\tau} = 0$$

$$\frac{d\lambda}{d\tau}\left[3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau}(2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}\right] = (q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{\left[3\lambda^2 + 2p_1\lambda + p_2 + e^{-\lambda\tau}(2q_1\lambda + q_2) - (q_1\lambda^2 + q_2\lambda + q_3)\pi e^{-\lambda\tau}\right]}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda e^{-\lambda\tau}} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda} - \frac{\tau}{\lambda}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2}{-\lambda(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3)} + \frac{(2q_1\lambda + q_2)}{(q_1\lambda^2 + q_2\lambda + q_3)\lambda} - \frac{\tau}{\lambda}$$

Put $\lambda = i\omega$ in the above we get

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{1}{\omega_0}\left[\frac{-3\omega_0{}^2 + 2ip_1\omega_0 + p_2}{(-\omega_0{}^3 + p_2\omega_0 + i(p_1\omega_0^2 - p_3))^2} + \frac{(2iq_1\omega_0 + q_2)}{-q_2\omega_0 + i(q_3 - q_1\omega_0^2)} + \tau i\right]$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{1}{\omega_0}\left[\frac{(-3\omega_0{}^2 + 2ip_1\omega_0 + p_2)((-\omega_0{}^3 + p_2\omega_0 - i(p_1\omega_0^2 - p_3)}{(-\omega_0{}^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2} + \frac{(2iq_1\omega_0 + q_2)(-q_2\omega_0 - i(q_3 - q_1\omega_0^2)}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2} + \tau i\right]$$

Real part of $\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{1}{\omega_0}\left[\frac{(-3\omega_0{}^2+p_2)(-\omega_0{}^3+p_2\omega_0)+2p_1\omega_0(p_1\omega_0^2-p_3)}{(-\omega_0{}^3+p_2\omega_0)^2+(p_1\omega_0^2-p_3)^2} + \frac{-q_2{}^2\omega_0+2q_1\omega_0(q_3-q_1\omega_0^2)}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}\right]$

$$(-\omega_0{}^3 + p_2\omega_0)^2 + (p_1\omega_0^2 - p_3)^2 = (q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2$$

$$= \frac{1}{\omega_0}\left[\frac{3\omega_0^5+\omega_0^3(2p_1^2-p_2-3p_2-2q_1^2)+(p_2^2-2p_1p_3+2q_1q_2-q_2^2)\omega_0}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}\right]$$

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1} = \left[\frac{3\omega_0^4+\omega_0^2(2p_1^2-4p_2-2q_1^2)+p_2^2-2p_1p_3+2q_1q_2-q_2^2}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}\right]$$

$$\left[\frac{d}{d\tau}Re(\lambda)\right] = \left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} = \left[\frac{3\omega_0^4 + \omega_0^2(2p_1^2 - 4p_2 - 2q_1^2) + p_2^2 - 2p_1p_3 + 2q_1q_3 - q_2^2}{(q_2\omega_0)^2 + (q_3 - q_1\omega_0^2)^2}\right]$$

$$\left[\frac{d}{d\tau}\mathrm{Re}(\lambda)\right] > 0$$

By using this condition $N_1>0$, $N_2 > 0$, $N_3>0$ we have $\left[\frac{d}{d\tau}(Re(\lambda))\right]_{\lambda=i\omega_0} > 0$

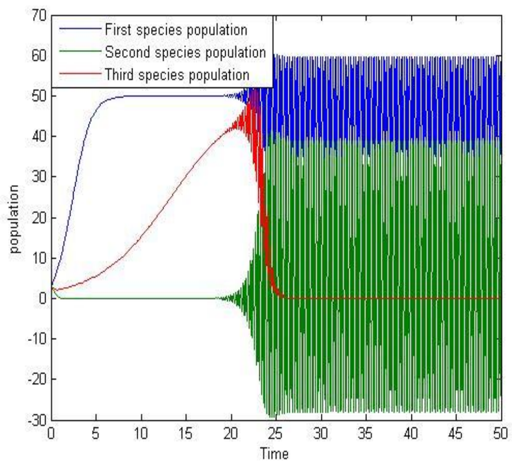
Therefore, the Hopf bifurcation occurs at $\tau > \tau_0$

6. Numerical Simulation

We study the Hopf bifurcations for the system (2.1) with the tolerance parameter (τ). For the system of equations, the parameters are identified as shown in the example 1. For different values of τ the graphs are shown below.

Example: 6.1 let us choose the following parameters for examination

$a_1 = 1, a_2 = 1, a_3 = 1, \alpha_{12} = 0.2, \alpha_{21} = 0.5, \alpha_{32} = 0.4, k_1 = 50, k_2 = 50, k_3 = 50\ , x = 3, y = 3, z = 3.$



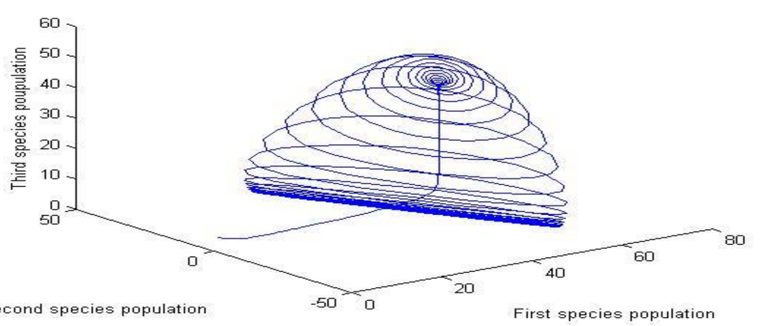


Fig. 6.1 (A) **Fig. 6.1 (B)**

The unbounded periodic solutions for the system (2.1) when τ = 0.069

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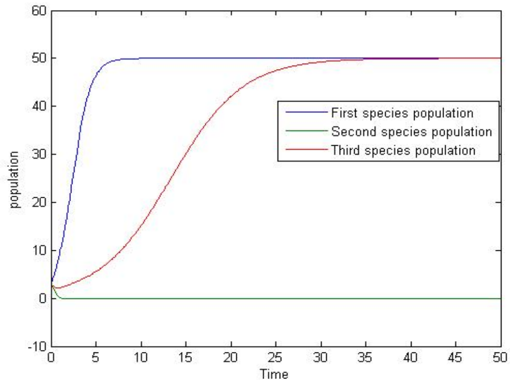


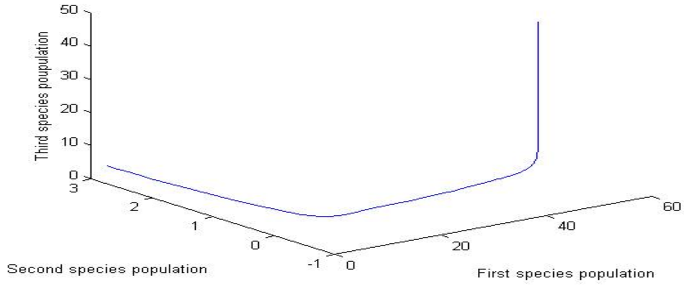
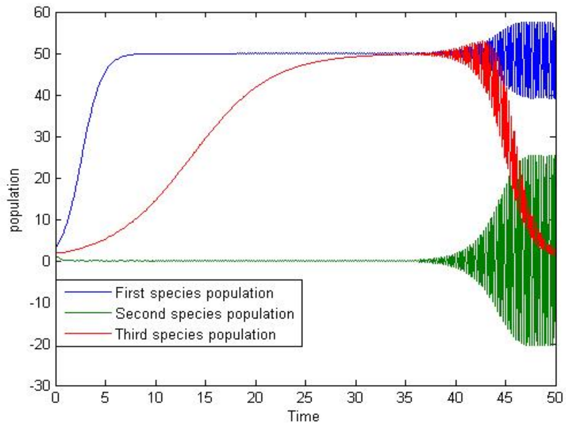


Fig. 6.1 (C) **Fig. 6.1 (D)**

The bounded solutions for the system (2.1) when τ = 0.068

Example: 6.2 let us choose the following parameters for examination

$a_1 = 1, a_2 = 1, a_3 = 1, \alpha_{12} = 0.1, \alpha_{21} = 0.5, \alpha_{32} = 0.4, k_1 = 50, k_2 = 50, k_3 = 50\ , x = 3, y = 1, z = 2.$



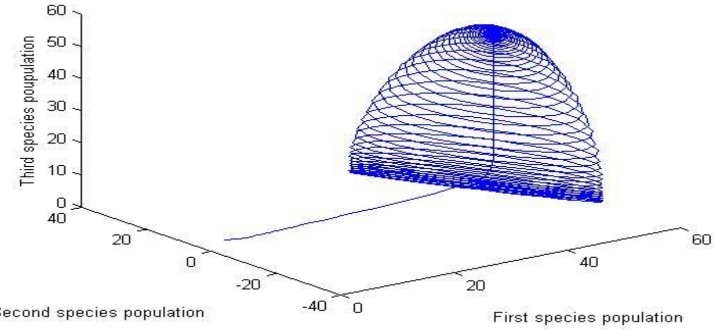
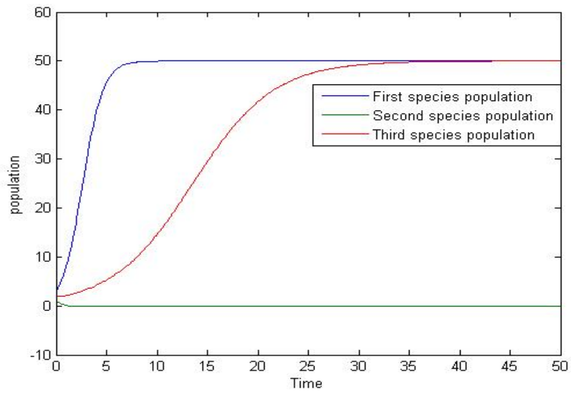


Fig. 6.2 (A) **Fig. 6.2 (B)**

The unbounded periodic solutions for the system (2.1) when τ = 0.065



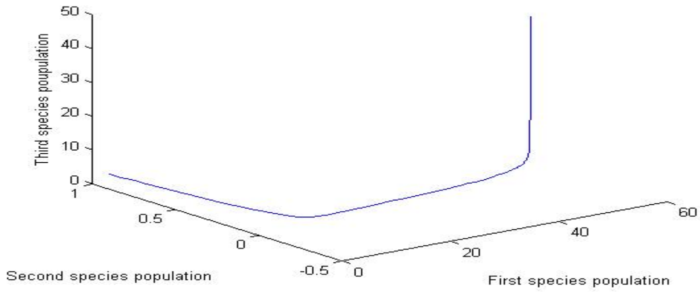


Fig. 6.2 (C) **Fig. 6.2 (D)**

The bounded solutions for the system (2.1) when τ = 0.064

Example: 6.3 Let us choose the following parameters for examination

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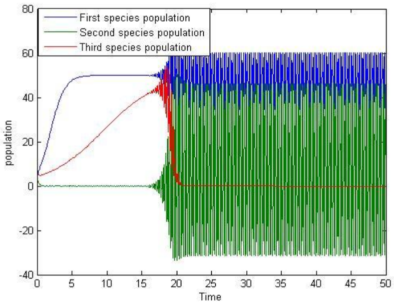
$a_1 = 1, a_2 = 0.5, a_3 = 0.25, \alpha_{12} = 0.1, \alpha_{21} = 0.5, \alpha_{32} = 0.4, k_1 = 50, k_2 = 50, k_3 = 50$, x=5,y=3,z=6.



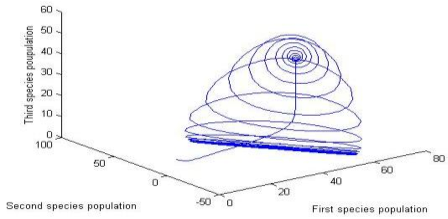
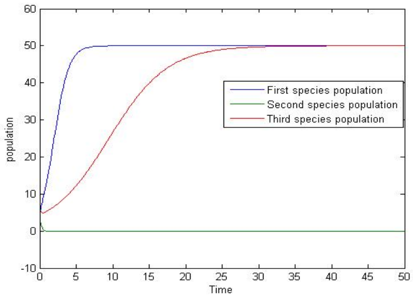


Fig. 6.3 (A) **Fig. 6.3 (B)**

The unbounded periodic solutions for the system (2.1) when τ = 0.072



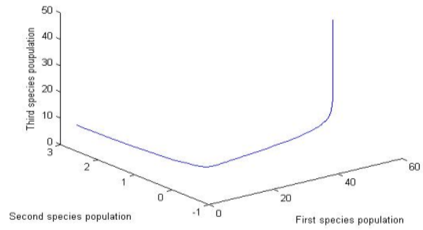
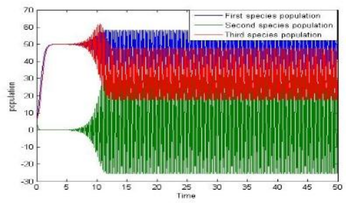


Fig. 6.3 (C) **Fig. 6.3 (D)**

The bounded solutions for the system (2.1) when τ = 0.071

Example: 6.4 Let us choose the following parameters for examination $a_1 = 3, a_2 = 2;, a_3 = 3, \alpha_{12} = 0.2, \alpha_{21} = 0.5, \alpha_{32} = 0.4, k_1 = 50, k_2 = 50, k_3 = 50,$

$$x = 5, y = 3, z = 6.$$



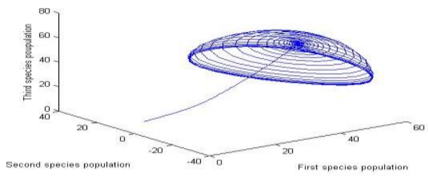
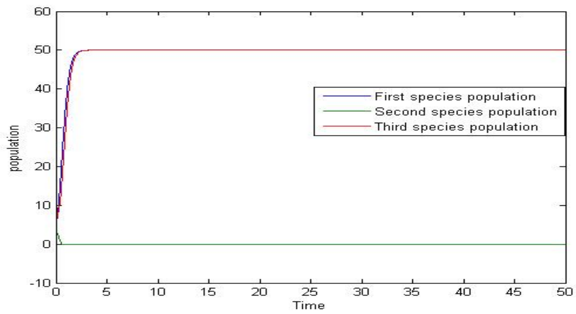


Fig. 6.4 (A) **Fig. 6.4 (B)**

The unbounded periodic solutions for the system (2.1) when τ = 0.65



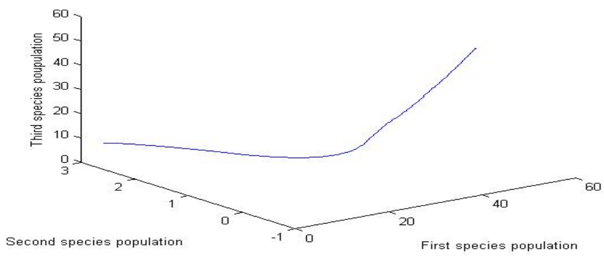


Fig. 6.4 (C) **Fig. 6.4 (D)**

The bounded solutions for the system (2.1) when $\tau = 0.6$

CONCLUSION

A logistic growth model involving three interacting species is considered for investigation. The species are denoted by x, y, and z. A time-delay parameter (τ) is incorporated into the interaction between the first species (x) and the second species (y). The coexistence equilibrium of the system is shown to be locally asymptotically stable in the absence of delay. Numerical simulations are performed for different values of (τ) the corresponding dynamical behaviours are illustrated through suitable examples. In this study, we examine the Hopf bifurcation that arises when the delay parameter(τ) is used as a bifurcation parameter. Four sets of numerical examples are considered to analyze the bifurcation characteristics. For each example, the system dynamics are observed for varying values of (τ). Based on these observations, it is found that Hopf bifurcation occurs in three out of the four examples, as summarized in the following table.

S. No	Example	Critical value
1	Example 6.1	$\tau > 0.068$
2	Example 6.2	$\tau > 0.064$
3	Example 6.3	$\tau > 0.071$
4	Example 6.4	$\tau > 0.6$

Table (7.1)

Hence the delay parameter τ stabilizes the system.

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