Coefficient Inequalities for Certain Univalent Analytic Starlike And Convex Functions in Leaf Like Domain Through Jackson Q-Derivative Operator

[1] Sanjay Issar, [2] Hemlata

[1] Department of Mathematics, Government Dungar College, Bikaner-334001, India

[2] Department of Mathematics, Tantia University, Sriganganagar-335002, India

ABSTRACT

Two subclasses of starlike and convex functions analytic in the unit open disk using q-derivative operator have been investigated in the present paper. The necessary and sufficient condition for the function belonging to these classes have been obtained. We further examine various properties, such as the Hadamard product and the quasi-Hadamard product. The coefficient estimates for the function belonging to these classes are also found.

Mathematics Subject Classification 2020: 30C45, 30C50

Keywords: Starlike, Convex, Hadamard product (or convolution), quasi-Hadamard product, Subordination, q-derivative operator.

INTRODUCTION

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

which are analytic in the unit open disc U = $\{z \in C : |z| < 1\}$. Let S be the sub class of A consisting of univalent functions. Let Ω be the class of analytic functions ω satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for all $z \in$ U.

Let f and g be two analytic functions in U, then function f is said to be subordinate to g if there exists an analytic function $w \in \Omega$ such that $f(z) = g(w(z))$ ($z \in$ U). We denote this subordination by $f \prec g$. In particular, if the function g is univalent in U, then the above subordination is equivalent to $f(0) = g(0)$ *and* $f($U$) \subseteq g($U$)$.

Quantum calculus, often referred to as q-calculus, offers a natural extension of classical calculus by incorporating a parameter q, where $0 < q < 1$ and without the notion of limit, thereby broadening the range of traditional analytical methods. In recent years, the field of q-calculus has drawn significant attention from researchers due to its strong connections with physics, quantum mechanics, and Geometric Function Theory (GFT). The application of q-calculus was initiated by Jackson ([13], [14]), who first investigated its uses and effectively introduced the q-derivative and q-integral operators.

Recently in the field of Geometric Function Theory many function classes have been introduced with the help of q-derivative and q-integral operators and investigated by a number of researchers including ([1], [2],[3],[6],[38], [26], [29], [34],[35] ,[36],[37],[39],).

The purpose of this article is to introduce and study two subclasses of univalent functions by applying q-derivative operator in conjunction with the principle of subordination.

Jackson q-derivative with $0 < q < 1$, of a function f is defined as ([11],[13],[14]).

$$(z \neq 0) \qquad D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \\ f' & (0) \qquad (z = 0), \end{cases}$$

provided that $f'(0)$ exists.

For f given by (1.1), we can easily obtain that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}.$$

It is easy to see that as $q \to 1^-$, $[n]_q \to n$ and $D_q f(z) = f'(z)$.

Also, we have the following q-derivative rules

$$D_q (f(z).g(z)) = g(z) D_q(f(z)) + f(qz) D_q(g(z))$$

$$D_q \frac{f(z)}{g(z)} = \frac{g(z) D_q f(z) - f(z) D_q g(z)}{g(z) g(qz)}$$

The $q - integral$ of the function f is defined as (see [13])

$$\int_0^z f(t) d_q t = (1-q)z \sum_{n=0}^{\infty} q^n f(zq^n), \qquad (1.2)$$

provided that the series converges. Here we observe that

$$\int_0^z t^n d_q t = \frac{z^{n+1}}{[n+1]_q} \quad (n \neq -1)$$

and

$$\int_0^z t^n d_q t = \frac{z^{n+1}}{n+1} \quad (n \neq -1).$$

as $q \to 1^-$.

A function $f \in A$ is starlike if $f(U)$ is starlike with respect to origin and it is convex if $f(U)$ is convex. Analytically in terms of subordination, a function $f \in A$ is starlike and convex if and only if the subordination relations $\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$

and $\frac{zf''(z)}{f'(z)} \prec \frac{2z}{1-z}$ for $z \in U$ respectively hold. Ma and Minda[21] introduced and studied the following two subclasses S*(ϕ) and K(ϕ) of starlike and convex functions respectively in term of subordination relations:

$$S^*(\phi) = \{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \}$$

and

$$\mathcal{K}(\phi) = \{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)\},$$

$'$ where ϕ is analytic and normalized by $\phi(0) = 1$ and ϕ (0) > 0 with $\Re\phi(z) > 0$ in U. Seoudy and Aouf [29] further generalized these classes by using the q-derivative operator in the following manner:
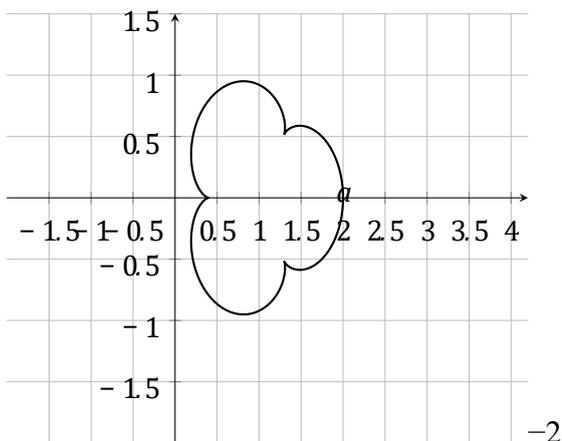
$$\mathcal{S}_q^*(\phi) = \{f \in \mathcal{A} : \frac{zD_qf(z)}{f(z)} \prec \phi(z)\}$$

and

$$\mathcal{K}_q(\phi) = \{f \in \mathcal{A} : \frac{D_q(zD_qf(z))}{D_qf(z)} \prec \phi(z)\}.$$

Recently, several Ma - Minda type subclasses of starlike and convex functions have been introduced and studied by considering different image domains $\phi(U)$. For some examples, we can see([7],[15], [17],[30],[31],[18],[32],[41]). Motivated by the work of Kumar and C‚etinkaya [17] , in the present paper we consider two subclasses

$\mathcal{S}_q^*(\psi)$ and K$_q(\psi)$ of starlike and convex functions respectively, which are associated with the analytic function $\psi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$.

The function ψ maps an open unit disc onto a simply connected bounded region in the right - half of the complex plane. Analytically these classes are defined as:

$$(1.3) \qquad \mathcal{S}_q^*(\psi) = \{f \in \mathcal{A} : \frac{zD_qf(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4\}$$

and

$$\mathcal{K}_q(\psi) = \{f \in \mathcal{A} : \frac{D_q(zD_qf(z))}{D_qf(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4\} \qquad . \ (1.4)$$

From (1.3) and (1.4), we find that

$$f(z) \in \mathcal{S}_q^*(\psi) \iff \int_0^z \frac{f(t)}{t} d_qt \in \mathcal{K}_q(\psi). \qquad (1.5)$$

Let f and g be two analytic functions of the form

$$f(z) = a_1 z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) of f and g is defined as

$$f(z) * g(z) = (f * g)(z) = a_1 b_1 z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

By taking $g(z)$ to be $\frac{z}{(1-z)}$ and $\frac{z}{(1-z)^2}$, it is easy to see that

$$f(z) * \frac{z}{1-z} = f(z), \tag{1.6}$$

$$f(z) * \frac{z}{(1-z)^2} = f'(z), \tag{1.7}$$

proceeding as above, we can obtain the following

$$zD_q f(z) = zD_q f(z) * \frac{z}{(1-z)} = f(z) * \frac{z}{(1-z)(1-qz)} \tag{1.8}$$

and

$$zD_q(zD_q f(z)) = zD_q f(z) * \frac{z}{(1-z)(1-qz)}. \tag{1.9}$$

Further, let T be the subclass of analytic functions with negative coefficients of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \quad (a_1 > 0, a_n \geq 0) \tag{1.10}$$

defined in U. For the function f defined by (1.10) and $g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n$ $(b_1 >$

$0, b_n \geq 0)$, the quasi -Hadamard product of f and g is defined as

$$f(z) *' g(z) = (f *' g)(z) = a_1 b_1 z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.11}$$

As above, we can define a quasi-Hadamard product of more than two functions. The quasi-Hadamard product of two or more functions has recently been defined and used by Owa ([23],-[24]), Kumar([19]-[20]), Sekine [28], Aouf [4], Frasin and Aouf [10], Hossen [12], Darwish [8] and El-Ashwah et al. [9].

In the next section we give characterizations for defined subclasses of q-starlike and q-convex functions with the help of Hadamard products. For each of these subclasses, we first find a function g depending on each concerning class $\mathcal{S}_q^*(\psi)$ and

$K_q(\psi)$, such that $\frac{1}{z}(f * g) \neq 0$ is both necessary and sufficient for f to be in $\mathcal{S}_q^*(\psi)$

and $K_q(\psi)$. Further we use these findings to determine the coefficients estimates for a function belonging to $\mathcal{S}_q^*(\psi)$ and $K_q(\psi)$. For some recent similar studies on various classes of analytic functions, one can find in ([5], [16],,[22],[25],[27], [29], [33] , [37], [40]) and the references cited therein.

Then we establish certain results concerning the quasi-Hadamard product of functions belonging in the classes $\mathcal{S}_q^*(\psi)$ and $K_q(\psi)$ analogous to the results due to Kumar ([19],[20]) and Sekine [28].

Hadamard product properties

We assume throughout this paper that $0 < q < 1$, and $\theta \in [0,2\pi)$.

Theorem 1. *The function $f \in$ T defined by* (1.10) *is in the class* $K_q(\psi)$ *if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1-(1+q)L]qz^2}{(1-z)(1-qz)(1-q^2z)}\right]' = 0 \qquad (2.1)$$

for all $L = \frac{5+4e^{\iota\theta}+e^{4\iota\theta}}{4e^{\iota\theta}+e^{4\iota\theta}}$, *where $\theta \in [0,2\pi)$ and is also true for* $L = 1$.

Proof. Suppose that the function $f \in K_q(\psi)$, then we have

$$\frac{D_q(zD_qf(z))}{D_qf(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \qquad (2.2)$$

Due to analyticity of the function $\frac{D_q(zD_qf(z))}{D_qf(z)}$ in $\mathbb{U}$, we have $D_qf(z) \neq 0$ which is equivalent to the fact that (2.1) holds for $L = 1$. In view of (2.2)

$$\frac{D_q(zD_qf(z))}{D_qf(z)} = 1 + \frac{4}{5}\omega(z) + \frac{1}{5}\omega^4(z)$$

where $\omega \in \Omega$ is a Schwarz function, hence

$$\frac{1}{z}\left[5zD_q(zD_qf(z)) - (5 + 4e^{4\iota\theta} + e^{4\iota\theta})zD_qf(z)\right] \neq 0. \qquad (2.3)$$

Using (1.8) and (1.9), we obtain

$$\frac{1}{z}\left[zD_qf(z) * \frac{5z}{(1-z)(1-qz)} - (5 + 4e^{\iota\theta} + e^{4\iota\theta})zD_qf(z) * \frac{z}{1-z}\right] \neq 0$$

or

$$-\frac{(4e^{\iota\theta} + e^{4\iota\theta})}{z}\left[zD_qf(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \neq 0,$$

where L is given by (2.1). Using the identity $zD_qf(z).g(z) = f(z).zD_qg(z)$, we get

$$\frac{1}{z}\left[f(z) * \frac{z + (1-(q+1)L)qz^2}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0,$$

which leads to (2.1).

For only if part of the theorem, suppose that f satisfies the condition (2.1). Since it is shown here that assumption (2.1) is equivalent to (2.3), so we have

$$\frac{D_q(zD_qf(z))}{D_qf(z)} \neq 1 + \frac{4}{5}e^{\iota\theta} + \frac{1}{5}e^{4\iota\theta}. \qquad (2.4)$$

Suppose that $\phi(z) = \frac{D_q(zD_qf(z))}{D_qf(z)}$ and $\psi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$. The relation (2.4) means that $\phi(U) \cap \psi(\partial U) = \emptyset$.

Thus, the simply connected domain $\phi(U)$ is included in a connected component of $\mathbb{C}\backslash\psi(\partial U)$. Therefore, using the fact that $\phi(0) = \psi(0)$ together with the univalence of the function $\psi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$, it follows that $\phi(z) \prec \psi(z)$. Hence $f \in K_q(\psi)$, which complete the proof of Theorem 1.

$\square$

By using the technique as given in Srivastava and Zayed [37], we prove the following convolution condition for the subclass $\mathcal{S}_q^*(\psi)$.

Theorem 2. *The function $f \in T$ defined by (1.10) is in the class $\mathcal{S}_q^*(\psi)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \diagup = 0 \qquad (2.5)$$

for all $L = \frac{5+4e^{\iota\theta}+e^{4\iota\theta}}{4e^{\iota\theta}+e^{4\iota\theta}}$, where $\theta \in [0,2\pi)$ and is also true for $L = 1$.

Proof. It follows from (1.5) that

$$f(z) \in \mathcal{S}_q^*(\psi) \text{ if and only if} \int_0^z \frac{f(t)}{t}d_qt \in \mathcal{K}_q(\psi).$$

Then according to the Theorem 1, the function f belongs to $\mathcal{S}_q^*(\psi)$ if and only if

$$(2.6) \qquad \frac{1}{z}\left[\left(\int_0^z \frac{f(t)}{t}d_qt\right) * g(z)\right] \neq 0,$$

where

$$g(z) = \frac{z + [1 - (1+q)L]qz^2}{(1-z)(1-qz)(1-q^2z)}.$$

From (1.2), we have

$$\int_0^z \frac{g(t)}{t}d_qt = \int_0^z \frac{1 + [1 - (1+q)L]qt}{(1-t)(1-qt)(1-q^2t)}d_qt$$

$$= z(1-q)\sum_{n=0}^{\infty} \frac{q^n + [1 - (1+q)L]zq^{2n+1}}{(1-zq^n)(1-zq^{n+1})(1-zq^{n+2})},$$

and therefore

$$\int_0^z \frac{g(t)}{t}d_qt = \frac{z - Lqz^2}{(1-z)(1-qz)}. \qquad (2.7)$$

Using the identity

$$\left(\int_0^z \frac{f(t)}{t}d_qt\right) * g(z) = f(z) * \left(\int_0^z \frac{g(t)}{t}d_qt\right)$$

and (2.7) in relation (2.6), we get the desired result (2.5). $\square$

Theorem 3. *The function $f \in T$ defined by (1.10) is in the class $K_q(\psi)$ if and only if*

$$a_1 - \sum_{n=2}^{\infty}[n]_q\left(1 - \frac{5q[n-1]_q}{4e^{\iota\theta}+e^{4\iota\theta}}\right)a_nz^{n-1} \diagup = 0 \qquad (2.8)$$

Proof. If $f \in$ T given by (1.10), then from Theorem 1, we have $f \in K_q(\psi)$ if and only if (2.1) holds. Since

$$\frac{1}{(1-z)(1-qz)(1-q^2z)} = 1+(1+q+q^2)z+(1+q+2q^2+q^3+q^4)z^2+(1+q+2q^2+2q^3+2q^4+q^5+q^6)z^3+...,$$

it follows that

$$\frac{z + [1 - (1+q)L]qz^2}{(1-z)(1-qz)(1-q^2z)} = z + \sum_{n=2}^{\infty}[n]_q\big(1 - q(L-1)[n-1]_q\big)z^n,$$

where $L = \frac{5+4e^{\iota\theta}+e^{4\iota\theta}}{4e^{\iota\theta}+e^{4\iota\theta}}$ and so (2.1) may be written as

$$a_1 - \sum_{n=2}^{\infty}[n]_q\Big(1 - \frac{5q[n-1]_q}{4e^{\iota\theta}+e^{\iota\theta}}\Big)a_n z^{n-1} \neq 0.$$

which completes the proof. □

Theorem 4. *The function $f \in$ T defined by (1.10) is in the class $\mathcal{S}_q^*(\psi)$ if and only if*

$$a_1 - \sum_{n=2}^{\infty}\big(1 - \frac{5q[n-1]_q}{4e^{\iota\theta}+e^{4\iota\theta}}\big)a_n z^{n-1} \Big/ = 0 \qquad (2.9)$$

Proof. Since

$$\frac{z}{(1-z)(1-qz)} = z + \sum_{n=2}^{\infty}[n]_q z^n$$

so

$$\frac{z - Lqz^2}{(1-z)(1-qz)} = z + \sum_{n=2}^{\infty}[1 - q(L-1)[n-1]_q]z^n \qquad (2.10)$$

In view of (2.5) and (2.10), a simple calculation provides the desired result. □**Theorem 5.** *If the function $f \in$ T defined by (1.10) satisfies the inequality*

$$\sum_{n=2}^{\infty}[n]_q\big(5[n]_q - 2\big)a_n \leq 3a_1 \qquad (a_n \geq 0, a_1 > 0), \qquad (2.11)$$

then $f \in K_q(\psi)$.

Proof. Since

$$\left|a_1 - \sum_{n=2}^{\infty}[n]_q\left(1 - \frac{5q[n-1]_q}{4e^{\iota\theta}+e^{4\iota\theta}}\right)a_n z^{n-1}\right|$$

$$\geq a_1 - \sum_{n=2}^{\infty}[n]_q\left|\left(1 - \frac{5q[n-1]_q}{4e^{\iota\theta}+e^{4\iota\theta}}\right)\right|a_n$$

$$\geq a_1 - \sum_{n=2}^{\infty}[n]_q\left|\left(1 + \frac{5q[n-1]_q}{3}\right)\right|a_n > 0$$

Thus, the inequality (2.11) holds and our result follows from Theorem 3. □

Using similar arguments to those in the proof of Theorem 5, we may also prove the Theorem 6.

Theorem 6. *If the function $f \in \mathrm{T}$ defined by* (1.10) *satisfies the inequality*

$$\sum_{n=2}^{\infty}(5[n]_q - 2)a_n \le 3a_1 \qquad (a_n \ge 0, a_1 > 0), \qquad (2.12)$$

then $f \in \mathcal{S}_q^*(\psi)$.

Quasi-Hadamard product properties

The quasi-Hadamard product properties of the classes $\mathcal{S}_q^*(\psi)$ and $K_q(\psi)$ are obtained in this section. For this purpose we need to define the following subclass of analytic functions:

A function f of the form (1.10) belongs to the class $\mathcal{S}_q^{(c)}(\psi)$ if and only if

$$\sum_{n=2}^{\infty}([n]_q^c(5[n]_q - 2)a_n \le 3a_1, \qquad (3.1)$$

where c is a non-negative real number. We note that for a non-negative real number c the class $\mathcal{S}_q^{(c)}(\psi)$ is non-empty as the function of the form

$$f_1(z) = a_1 z - \sum_{n=2}^{\infty} \frac{3a_1}{([n]_q)^c(5[n]_q - 2)}\lambda_n z^n,$$

where $a_1 > 0$, $\lambda_n > 0$ and $\sum_{n=2}^{\infty}\lambda_n \le 1$ satisfies the inequality (3.1). Here we note that $\mathcal{S}_q^{(1)}(\psi) \equiv \mathcal{K}_q(\psi)$ and for $c = 0$, $\mathcal{S}_q^{(0)}(\psi) \equiv \mathcal{S}_q^*(\psi)$. Further, $\mathcal{S}_q^{(c)}(\psi) \subset \mathcal{S}_q^{(n)}(\psi)$ if $c > n \ge 0$, the containment being proper.

Let the functions of the form

$$f_i(z) = a_{1,i} - \sum_{n=2}^{\infty} a_{n,i}z^n \qquad (a_{1,i} > 0, a_{n,i} \ge 0) \qquad (3.2)$$

and

$$\sum_{n=2}^{\infty} g_i(z) = b_{1,i} - \sum b_{n,i}z^n \qquad (b_{1,i} > 0, b_{n,i} \ge 0) \qquad (3.3)$$

be analytic in the unit disc U.

Theorem 7. *Let the functions $f_i(i = 1,2,\cdots,m)$ given by* (3.2), *belong to the class* $\mathcal{S}_q^*(\psi)$. *Then, the quasi-Hadamard product $f_1 *' f_2 *' \cdots *' f_m$ belongs to the class* $\mathcal{S}_q^{(m-1)}(\psi)$.

Proof. Here, we need to show that

$$\sum_{n=2}^{\infty} \left[\left([n]_q\right)^{m-1} \left(5[n]_q - 2\right) \prod_{i=1}^{m} a_{n,i} \right] \leq 3 \prod_{i=1}^{m} a_{1,i} \tag{3.4}$$

Since $f_i(z) \in \mathcal{S}_q^*(\psi)$, we have

$$\sum_{n=2}^{\infty} (5[n]q - 2)a_{n,i} \leq 3a_{1,i} \qquad (a_{1,i} > 0, \ a_{n,i} \geq 0). \tag{3.5}$$

Therefore

$$a_{n,i} \leq \frac{3}{5[n]_q - 2} a_{1,i},$$

which implies

$$a_{n,i} \leq \frac{1}{[n]_q} a_{1,i}, \tag{3.6}$$

Using (3.6) for $i = 1,2,\cdots,m-1$ and (3.5) for $i = m$, we get

$$\sum_{n=2}^{\infty} \left[\left([n]q\right)^{m-1} (5[n]q - 2) \prod_{i=1}^{m} a_{n,i} \right] = \sum_{n=2}^{\infty} \left[\left(\prod_{i=1}^{m-1} [n]q\, a_{n,i} \right) (5[n]q - 2) a_{n,m} \right]$$

$$\leq \left(\prod_{i=1}^{m-1} a_{1,i} \right) \sum_{n=2}^{\infty} (5[n]q - 2) a_{n,m}$$

$$\leq 3 \prod_{i=1}^{m} a_{1,i}.$$

which completes the proof. □

Theorem 8. *Let the functions $f_i(i = 1,2,\cdots,m)$ given by (3.2), belong to the the class $K_q(\psi)$. Then, the quasi-Hadamard product $f_1 *' f_2 *' \cdots *' f_m$ belongs to the class $S_q(2m-1)(\psi)$.*

Proof. Since $f_i(z) \in K_q(\psi)$, we have

$$\sum_{n=2}^{\infty} [n]_q \left(5[n]_q - 2\right) a_{n,i} \leq 3a_{1,i} \qquad (a_{1,i} > 0, \ a_{n,i} \geq 0). \tag{3.7}$$

Therefore

$$a_{n,i} \leq \frac{3}{[n]_q (5[n]_q - 2)} a_{1,i},$$

which implies

$$a_{n,i} \leq \frac{1}{([n]_q)^2} a_{1,i}, \tag{3.8}$$

Using (3.8) for $i = 1,2,\cdots,m-1$ and (3.7) for $i = m$ in the following:

$$\sum_{n=2}^{\infty} \left[([n]q)2m-1(5[n]q - 2)\prod_{i=1}^{m} an,i \right] = \sum_{n=2}^{\infty} \left[\left\{ \prod_{i=1}^{m-1} ([n]q)2an,i \right\} [n]q(5[n]q - 2)an,m \right]$$

$$\leq \left(\prod_{i=1}^{m-1} a1,i \right) \sum_{n=2}^{\infty} [n]q(5[n]q - 2)an,m$$

$$\leq 3 \prod_{i=1}^{m} a_{1,i}.$$

which is the required condition for $f_1 *' f_2 *' \cdots *' f_m$ to be in the class $S_q^{(2m-1)}(\psi)$.

$\square$

On using the similar arguments as used in the Theorem 7 and Theorem 8 we may obtain the following result:

Theorem 9. *Let the functions $f_i(i = 1,2,\cdots,m)$ given by (3.2), belong to the the class $K_q(\psi)$ and the functions $g_j(j = 1,2,\cdots,s)$ given by (3.3) belong to the class*

$\mathcal{S}_q^*(\psi)$. *Then, the quasi-Hadamard product $f_1 *' f_2 *' \cdots *' f_m *' g_1 *' g_2 *' \cdots *' g_s$ belongs to the class $\mathcal{S}_q^{(2m+s-1)}(\psi)$*

CONCLUSIONS

In this paper, we have used q-calculus to introduce two subclasses $\mathcal{S}_q^*(\psi)$ and $K_q(\psi)$ which are q-analogue of the classes studied by Kumar and C¸etinkaya [17]. These classes are associated with the analytic function $\psi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$, which maps an open unit disc onto the leaf shaped bounded region. We investigate several fundamental aspects of the classes $\mathcal{S}_q^*(\psi)$ and $K_q(\psi)$, including Hadamard products, necessary and sufficient conditions, and coefficient estimates. Additionally, we establish new results concerning the quasi-Hadamard products associated with these classes.

REFERENCES

1. S. Agrawal and S. K. Sahoo. A generalization of starlike functions of order α. Hokkaido Mathematical Journal, 46(1):15–27, 2017. Zbl 1361.30017.
2. Abdullah Alsoboh, Ala Amourah, Omar Alnajar, Mamoon Ahmed, and Tamer M. Seoudy. Exploring q-fibonacci numbers in geometric function theory: Univalence and shell-like starlike curves. Mathematics, 13(8), 2025.
3. Abdullah Alsoboh and Georgia Irina Oros. A class of bi-univalent functions in a leaf-like domain defined through subordination via q-calculus. Mathematics, 12(10), 2024.
4. M. K. Aouf. The quasi-hadamard product of certain analytic functions. Applied Mathematics Letters, 21(11):1184–1187, 2008. MR2459845; Zbl 1159.30305.
5. M. K. Aouf, A. O. Mostafa, and H. M. Zayed. Convolution properties for some subclasses of meromorphic functions of complex order. Abstract and Applied Analysis, 2015:973613, 2015. Article ID 973613.
6. Mihai Aron. Coefficient estimates and fekete-szeg¨o problem for some classes of univalent functions generalized to a complex order. Demonstratio Mathematica, 58, 02 2025.

7. N. E. Cho, V. Kumar, S. S. Kumar, and V. Ravichandran. Radius problems for starlike functions associated with the sine function. Bulletin of the Iranian Mathematical Society, 45:213–232, 2019. Zbl 1409.30009.

8. H. E. Darwish. The quasi-hadamard product of certain starlike and convex functions. Applied Mathematics Letters, 20:692–695, 2007. Zbl 1111.30009.

9. R. M. El-Ashwah, M. K. Aouf, and N. Breaz. On quasi-hadamard products of p-valent functions with negative coefficients defined by using a differential operator. Acta Universitatis Apulensis. Mathematica Informatica, 28:61–70, 2011. Zbl 1274.30034.

10. B. A. Frasin and M. K. Aouf. Quasi-hadamard product of a generalized class of analytic and univalent functions. Applied Mathematics Letters, 23:347–350, 2010. Zbl 1186.30015.

11. G. Gasper and M. Rahman. Basic Hypergeometric Series. Cambridge University Press, Cambridge, 1990. Zbl 0695.33001.

12. H. M. Hossen. Quasi-hadamard product of certain p-valent functions. Demonstratio Mathematica, 33:277–281, 2000. Zbl 0963.30011.

13. F. H. Jackson. On q-definite integrals. Quarterly Journal, 41:193–203, 1910. JFM 41.0317.04.

14. F. H. Jackson. q-difference equations. American Journal of Mathematics, 32:305–314, 1910. MR1506108; JFM 41.0502.01.

15. W. Janowski. Extremal problems for a family of functions with positive real part and for some related families. Annales Polonici Mathematici, 23:159–177, 1970. Zbl 0199.39901.

16. S. Kant and P. P. Vyas. Convolution properties for certain subclasses of meromorphic p-valent functions. Journal of Fractional Calculus and Applications, 12(2):197–203, 2021. Zbl 1499.30102.

17. S. Kumar and A. C¸etinkaya. Coefficient inequalities for certain starlike and convex functions. Hacettepe Journal of Mathematics and Statistics, 51(1):156– 171, 2022. Zbl 1499.30108.

18. S. S. Kumar and G. Kamaljeet. A cardioid domain and starlike functions. Analysis and Mathematical Physics, 11(2):35, 2021. Zbl 1461.30042.

19. V. Kumar. Hadamard product of certain starlike functions. Journal of Mathematical Analysis and Applications, 110:425–428, 1985. Zbl 0585.30013.

20. V. Kumar. Quasi-hadamard product of certain univalent functions. Journal of Mathematical Analysis and Applications, 126:70–77, 1987. Zbl 0625.30014.

21. W. C. Ma and D. Minda. A unified treatment of some special classes of univalent functions. In Conference Proceedings, Lecture Notes in Analysis, volume 1, pages 157–169, 1994. Zbl 0823.30007.

22. O. Mostafa, M. K. Aouf, H. M. Zayed, and T. Bulboacˇa. Convolution conditions for subclasses of meromorphic functions of complex order associated with basic bessel functions. Journal of the Egyptian Mathematical Society, 25(3):286–290, 2017. Zbl 1376.30009.

23. S. Owa. On the classes of univalent functions with negative coefficients. Mathematica Japonica, 27:409–416, 1982. Zbl 0497.30015.

24. S. Owa. On the hadamard products of univalent functions. Tamkang Journal of Mathematics, 14:15–21, 1983. Zbl 0527.30001.

25. S. Ponnusamy. Convolution properties of some classes of meromorphic univalent functions. Proceedings of the Indian Academy of Sciences. Mathematical Sciences, 103(1):73–89, 1993. Zbl 0772.30017.

26. S. D. Purohit and R. K. Raina. Certain subclasses of analytic functions associated with fractional q-calculus operators. Mathematica Scandinavica, 109:55–70, 2011. Zbl 1229.33027.

27. V. Ravichandran, S. S. Kumar, and K. G. Subramanian. Convolution conditions for spirallikeness and convex spirallikeness of certain meromorphic p-valent functions. Journal of Inequalities in Pure and Applied Mathematics, 5(1):Article 7, 2004. Zbl 1062.30017.

28. T. Sekine. On quasi-hadamard products of p-valent functions with negative coefficients. In H. M. Srivastava and S. Owa, editors, Univalent Functions, Fractional Calculus, and Their Applications, pages 317–328. Ellis Horwood / John Wiley, Chichester, 1989. Zbl 0683.00012.

29. T. M. Seoudy and M. K. Aouf. Convolution properties for certain classes of analytic functions defined by q-derivative operator. Abstract and Applied Analysis, 2014:Article ID 846719 (7 pages), 2014. MR3272218; Zbl 07023188.

30. K. Sharma, N. K. Jain, and V. Ravichandran. Starlike functions associated with a cardioid. Africa Mathematica, 27:923–939, 2016. Zbl 1352.30015.

31. L. Shi, H. M. Srivastava, N. E. Cho, and M. Arif. Sharp coefficient bounds for a subclass of bounded turning functions with a cardioid domain. Axioms, 12(8):775, 2023. DOI: 10.3390/axioms12080775.

32. J. Sokł□o and J. Stankiewicz. Radius of convexity of some subclasses of strongly starlike functions. Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka, 147:101–105, 1996. Zbl 0880.30014.

33. Soni, P. P. Vyas, and S. Kant. Convolution properties for certain subclasses of meromorphic p-valent functions by means of cassinian ovals. Facta Universitatis. Series Mathematics and Informatics, 38(1):25–30, 2023. Zbl 07742875.

34. H. M. Srivastava. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. Iranian Journal of Science and Technology, Transactions A: Science, 44:327–344, 2020.

35. H. M. Srivastava, M. Arif, and M. Raza. Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator. AIMS Mathematics, 6(6):5869–5885, 2021. DOI: 10.3934/math.2021347.

36. H. M. Srivastava, M. Tahir, B. Khan, M. Darus, N. Khan, and Q. Ahmad. Certain subclasses of meromorphically q-starlike functions associated with the q-derivative operators. Ukrainian Mathematical Journal, 73:1462–1477, 2022. Zbl 1492.30049; DOI: 10.1007/s11253-022-02005-5.

37. R. Srivastava and H. M. Zayed. Subclasses of analytic functions of complex order defined by q-derivative operator. Studia Universitatis Babeș-Bolyai Mathematica, 64(1):71–80, 2019. MR3928603; Zbl 1438.30099.

38. Wahid Ullah, Sarfraz Nawaz Malik, Daniel Breaz, and Luminita-Ioana Cotirla. On logarithmic coefficients for starlike functions related to the nephroid domain. AIMS Mathematics, 10(11):26905–26925, 2025.

39. P. P. Vyas. Convolution conditions for certain subclasses of meromorphic pvalent functions. Facta Universitatis, Series Mathematics and Informatics, 37(1):159–168, 2022. Zbl 1499.32008.

40. P. P. Vyas. Convolution conditions for certain subclasses of meromorphic p-valent functions. Facta Universitatis. Series Mathematics and Informatics, 37(1):159–168, 2022. Zbl 1499.32008.

41. L. A. Wani and A. Swaminathan. Starlike and convex functions associated with a nephroid domain. Bulletin of the Malaysian Mathematical Sciences Society, 44:79–104, 2021. Zbl 1461.30046.