INTERNATIONAL JOURNAL OF LATEST TECHNOLOGY IN ENGINEERING, MANAGEMENT & APPLIED SCIENCE (IJLTEMAS)

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XV, Issue III, March 2026

Generalized Theorems of Fixed Point for Fuzzy Contractions in Fuzzy Metric Space

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DOI: https://doi.org/10.51583/IJLTEMAS.2026.150300039

Received: 16 March 2026; Accepted: 25 March 2026; Published: 09 April 2026

ABSTRACT

In this paper, we establish a generalized fixed-point theorem for fuzzy contractions in fuzzy metric spaces. The result extends the well-known Banach contraction principle into the setting of fuzzy metric spaces by employing a generalized fuzzy contraction. Examples are provided to demonstrate the applicability and generalization of classical fixed-point results. Fuzzy fixed-point techniques are used in mathematical modelling to solve problems where traditional methods fail due to imprecise or uncertain data. To obtain fuzzy fixed points, different contraction conditions are implemented in a fuzzy context.

Keywords: Fuzzy metric space, Fixed point theorem, Fuzzy contraction, Generalized fuzzy metric, Banach contraction

INTRODUCTION

The concept of fuzzy metric space was first introduced by Kramosil and Michalek (1975) as a generalization of the classical metric space to handle uncertainty and imprecision. Later, George and Veeramani (1994) modified the definition to make it suitable for topological analysis.

Fixed point theory, originally studied by Banach (1922), plays an essential role in nonlinear analysis and its applications. Extending fixed point theorems to fuzzy metric spaces has become an active area of research, as it integrates uncertainty with analytical rigor.

In this paper, we propose a generalized fuzzy contraction mapping and prove a fixed point theorem for such mappings in fuzzy metric spaces.

Preliminaries

Definition

A fuzzy metric space is a triple

$$(X, P, *)$$

where:

1. X is a non-empty set
2. $*$ is a continuous t-norm
3. $P: X \times X \times (0, \infty) \to [0,1]$

satisfying the following conditions:

1. $P(x, y, r) = 1$ if and only if $x = y$

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XV, Issue III, March 2026

2. $P(x,y,r) = P(y,x,r)$
3. $P(x,y,r) * P(y,z,s) \leq P(x,z,r+s)$
4. $P(x,y,r)$is non-decreasing in r
5. $\lim_{r\to 0} P(x,y,r) = 0$

Generalized Fuzzy Contraction

Definition

Let $(X,P,*)$be a fuzzy metric space. A mapping $T: X \to X$is called a **generalized fuzzy contraction** if there exists $k \in (0,1)$ such that for all $x,y \in X$and for all $r > 0$,

$$P(Tx,Ty,r) \geq P\left(x,y,\frac{r}{1-k}\right)$$

This condition generalizes the classical fuzzy contraction and allows a broader class of mappings to satisfy fixed point results.

Main Theorem (Results)

Generalized Fixed Point Theorem

Let $(X,P,*)$be a complete fuzzy metric space and $T: X \to X$ be a generalized fuzzy contraction.

Then Thas a unique fixed point $x^* \in X$, i.e.,

$$Tx^* = x^*$$

Proof

Let $x_0 \in X$be arbitrary and define a sequence $\{x_n\}$by:

$$x_{n+1} = Tx_n$$

Using the generalized contraction property,

$$P(x_{n+1},x_{n+2},r) = P(Tx_n,Tx_{n+1},r) \geq P\left(x_n,x_{n+1},\frac{r}{1-k}\right)$$

Similarly,

$$P(x_{n+2},x_{n+3},r) = P(Tx_{n+1},T_{x+2},\frac{r}{1-k}) \geq P\left(x_n,x_{n+1},\frac{r}{(1-k)^2}\right)$$

By induction,

$$P(x_n,x_{n+m},r) \geq P\left(x_0,x_1,\frac{r}{(1-k)^n}\right)$$

Since $0 < k < 1$,

$$(1-k)^n \to 0$$

as $n \to \infty$

Hence,

$$P(x_n,x_{n+1},r) \to 1$$

Thus, $\{x_n\}$is a Cauchy sequence in $(X,P,*)$.

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Now we have show completences of X

Because $X$is complete, the sequence x_n converges to some $x^* \in X$, i.e.,

$$lim_{n\to\infty} P(x_n, x^*, r) = 1 \text{for all } r > 0$$

Now we show that $x^*$is a fixed point of T

We have $x_{n+1} = Tx_n$.

Now consider:

$$P(Tx^*, x^*, r) \geq \quad P(Tx^*, Tx_n, \frac{r}{1-k}) \times P\left(Tx_n, x^*, \frac{r}{1-k}\right)$$

As $n \to \infty$, $x_n \to x^*$.

$Tx_n = x_{n+1} \ \to Tx^*$, and by

By continuity of P,

$$P(Tx^*, x^*, r) = 1.$$

Uniqueness of Fixed Point

Let $x^*$and y^*be two fixed points of T.

Let (X,P,*) be fuzzy metric stage and let T:X $\to$ X be a self-mapping satisfying the generalized fuzzy contraction condition

$$P(Tx, Ty, r) \geq \phi(P(\mathrm{x}, y, r))$$

where ϕ: $[0,1] \to [0,1]$is continuous and strictly increasing such that $\phi(s) > s$ for all $s \in (0,1)$. If T has a fixed point in X the it is unique.

Proof

Let x^* and y^* be two fixed points of T then

$$Tx^* = x^*, Ty^* = y^*$$

$$P(x^*, y^*, r) = P(Tx^*, Ty^*, r) \geq \phi(P(x^*, y^*, r))$$

Since $\phi(s) > s$ for all $s \in (0,1)$, the above inequality is possible only if

$$P(x^*, y^*, r) = 1 \quad r > 0.$$

By the properties of fuzzy metric space,

$$P(x^*, y^*, r) = 1 \ \ r > 0$$
$$\Rightarrow x^* = y^*.$$

Hence, the fixed point of $T$is unique.

Applications

Differential Equations

Many fuzzy differential and integral equations can be converted into fixed point problems in fuzzy metric spaces, ensuring the existence of solutions.

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Fuzzy metric spaces are used in pattern recognition and image processing fixed point results guarantee convergence of iterative algorithms.

CONCLUSION

In this paper, we proved a generalized fixed point theorem for contraction mappings in fuzzy metric spaces. The result provides a unifying framework for several well-known fixed point theorems and demonstrates that under suitable fuzzy contractive conditions, the mapping admits a unique fixed point.

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