Cartesian Product and Direct Product of Finite Prime Fuzzy BD-Ideals of BD-Algebras

Esraa Kareem Kadhim[1], Huda Qusay Hashim[2], Hanan Hayder Mohammed [3]

[1,3]Department of Mathematic, Faculty of Basic Education, University of Kufa, Najaf, Iraq

[2]Department of Mathematic, Faculty of Management Technical, University of AL-Furat AL-Awsat Technical, Kufa, Iraq

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ABSTRACT

Fuzzy Bd-ideals provide a useful framework for studying Bd-algebras. This paper focuses on the Cartesian and direct products of finite prime fuzzy Bd-ideals. We extend the idea of primeness to these constructions and analyze how it behaves under product operations. Key properties of the Cartesian product are obtained, including conditions that keep primeness intact. We also study the direct product case and identify when the prime structure is preserved. These findings give a clearer view of how fuzzy ideals interact within product structures in Bd-algebras.

Keywords: Bd-algebra, Cartesian Product prime fuzzy Bd-ideal, Cartesian Product semiprime fuzzy Bd-ideal, Direct Product of Finite Prime.

INTRODUCTION

In 2022, Bantaojai and colleagues [6] introduced a novel algebraic framework known as Bd-algebras by merging certain features of B-algebras and d-algebras. More recently, in 2024, Nakkhasen et al. [4] extended fuzzy set theory to Bd-algebras by defining fuzzy Bd-ideals and examining a range of their structural properties.In this work, we present the notions of prime and semiprime subsets, together with their fuzzy counterparts, within the setting of Bd-algebras. We further explore the connections between classical prime (and semiprime) Bd-ideals and their fuzzy analogues. In addition, we introduce the concept of the Cartesian product of prime and semiprime fuzzy Bd-ideals and investigate several related properties. Finally, we analyze the behavior of the direct product of finite prime and semiprime fuzzy Bd-ideals in Bd-algebras.

METHODOLOGY

This study uses a theoretical approach to investigate prime and semiprime fuzzy ideals in Bd-algebras. First, the main definitions of Bd-algebras and fuzzy sets are reviewed. Then, prime and semiprime fuzzy ideals are introduced and their basic properties are derived using algebraic methods.

After that, the Cartesian and direct products of finite fuzzy ideals are constructed and analyzed to study their behavior. Several results are proved to show how these properties are preserved under these operations. Finally, simple examples are given to support the theoretical results.

Preliminaries: This section presents several foundational results that support the demonstration of the principal theorem.

Definition [6]. An algebraic structure $(\aleph, *, 0)$ is called a Bd-algebra if $\aleph$ is a non-empty set, $*$ is a binary operation defined on $\aleph$, and 0 is a special element in, such that for all ɕ, ʐ∈ $\aleph$ the following conditions hold: ƀ

(i) ɕ $* 0 =$ ɕ;

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(ii) if ɕ ∗ ʐ = 0 and ʐ ∗ ɕ = 0, then ʐ = ɕ.

Definition [4]. Let I be a nonempty subset of a Bd-algebra (ℵ,∗,0). Then I is called a Bd-ideal of ℵ if it satisfies the following properties:

(i) 0 ∈ I;
(ii) if for all χ , 𝓛∈ ℵ, χ ∗ 𝓛 ∈ I and 𝓛∈ I, then χ ∈ I;
(iii) for every χ ∈ I and χ ∈ ℵ, χ ∗𝓛 ∈I.

Definition [4]. Let (ℵ,∗,0) be a Bd-algebra. A fuzzy set ɱ of ℵ is called a fuzzy Bd-ideal of ℵ if, for each χ ,𝓛 ∈ℵ it satisfies the following conditions:

(i) ɱ (0) ≥ ɱ (χ);
(ii) ɱ(χ) ≥ min{ɱ(χ ∗ 𝓛),ɱ(𝓛)};
(iii) ɱ(χ ∗ 𝓛) ≥ ɱ(χ).

Definition [2]. Let ℵ be a Bd-algebra and let 𝒜 be a nonempty subset of ℵ, Then.

i. 𝒜 is called prime, if for any χ ,𝓛∈ℵ, (χ ∗ 𝓛) ∈ 𝒜, then χ ∈ 𝒜 or 𝓛∈ 𝒜.
ii. 𝒜is called semiprime, if for any χ ∈ ℵ, (χ ∗χ) ∈ 𝒜, then χ ∈𝒜.

Definition [2]. Let ℵ be a Bd-algebra and let ɱ be a fuzzy set of ℵ. Then ɱ is said to be:

(i) prime if ɱ(χ ∗𝓛)≤ max{ɱ(χ),ɱ(𝓛)}, for all χ ,𝓛∈ℵ;
(ii) semiprime if ɱ(χ ∗ χ) ≤ ɱ(χ), for all χ ∈ ℵ. It is known that every prime fuzzy set of a Bd-algebra ℵ is also a semiprime fuzzy set of ℵ.

In general, the converse of this statement is not true as the following example shows:

Example [3]. let $ℵ = \{0, y, d, e\}$ with the binary operation (∗) on ℵ as follows:

∗	0	y	d	e
0	0	y	y	y
y	y	y	y	y
d	d	d	d	d
e	e	y	y	e

Then (ℵ,∗,0). is a Bd-algebra. ɱ: ℵ→[0,1] , ƀ:ℵ→[0,1]

The fuzzy sets ɱ and ƀ of ℵ defined by

ɱ (0) = 0.9, ɱ (y) = 0.7, ɱ(d) =ɱ(e) = 0.5,

and ƀ (0) = 0.8, ƀ (y) = 0.3, ƀ(d) = 0.4, ƀ (e) = 0.7

By routine calculate that *ɱ* is prime fuzzy set of ℵ. *ƀ* is semiprime but not prime is prime fuzzy set of ℵ, because ƀ $(y * d) >$ max{ ƀ (y), ƀ (d)}.

RESULTS

The outcomes of this study show that prime and semiprime fuzzy ideals in Bd-algebras largely retain their structural behavior when combined through Cartesian and direct product constructions, under appropriate conditions. The derived statements describe when these characteristics remain stable within product formations.

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Additionally, the illustrative cases demonstrate the validity of the theoretical results and help clarify how the concepts operate in a clearer and more intuitive manner.

Definition. Let $\mathcal{A} = \{(\chi, m_{\mathcal{A}}(\chi))|\chi \in \aleph\}$ and $\mathcal{B} = \{(\chi, m_{\mathcal{B}}(\chi))\,|\,\chi \in \aleph\}$

are two prime fuzzy Bd-ideals of $\aleph$, Then the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathcal{A} \times \mathcal{B} = (\aleph \times \aleph,\ m_{\mathcal{A}\times\mathcal{B}})$ such that $m_{\mathcal{A}\times\mathcal{B}}: \aleph \times \aleph \to [0,1]$ is defined by $(m_{\mathcal{A}\times\mathcal{B}})(\chi, \mathscr{L}) = \min\{m_{\mathcal{A}}(\chi),\ m_{\mathcal{B}}(\mathcal{L})\}$, for all $\chi, \mathscr{L} \in \aleph \times \aleph$.

Theorem. Let A $= \{(\chi, m_{\mathcal{A}}(\chi))|\chi \in \aleph\}$ and B $= \{(\chi, m_{\mathcal{B}}(\chi))\,|\,\chi \in \aleph\}$ be is prime fuzzy Bd-ideals of $\aleph$, then $\mathcal{A} \times \mathcal{B}$ is prime fuzzy Bd-ideals of $\aleph \times \aleph$.

Proof. For all $(\chi * \mathscr{L}) \in \aleph \times \aleph$, Now, let $(\chi_1, \chi_2), (\mathcal{L}_1, \mathcal{L}_2), \in \aleph \times \aleph$, then

$(m_{\mathcal{A}} \times m_{\mathcal{B}})(\chi_1, \chi_2) * (\mathcal{L}_1, \mathcal{L}_2)) = (m_{\mathcal{A}} \times m_{\mathcal{B}})(\chi_1 * \mathcal{L}_1, \chi_2 * \mathcal{L}_2)$

$\preccurlyeq$ r max $\{$r max $\{(m_{\mathcal{A}})(x_1), (m_{\mathcal{A}})(y_1)\}$, r max $\{(m_{\mathcal{B}})(\chi_2), (m_{\mathcal{B}})(\mathcal{L}_2)\}\}$

$=$ r max $\{$r max $\{(m_{\mathcal{A}} \times m_{\mathcal{B}})(\chi_1 * \chi_2)$, r max $\{(m_{\mathcal{A}} \times m_{\mathcal{B}})(\mathcal{L}_1 * \mathcal{L}_2)\}\}$

$=$ r max $\{(m_{\mathcal{A}} \times m_{\mathcal{B}})(\chi_1, \chi_2), (m_{\mathcal{A}} \times m_{\mathcal{B}})(\mathcal{L}_1, \mathcal{L}_2)\}$.

Hence $\mathcal{A} \times \mathcal{B}$ is prime fuzzy Bd-ideals of $\aleph \times \aleph$. ■

Definition. Let $\mathcal{A} = \left\{\left(\chi, m_{A}(\chi)\right)\middle|\chi \in \aleph\right\}$ and $\mathcal{B} = \left\{\left(\chi, m_{\mathcal{B}}(\chi)\right)\middle|\chi \in \aleph\right\}$ are prime fuzzy Bd-ideals of $\aleph$, For s $\in [0,1]$, the set $U(m_{\mathcal{A}} \times m_{\mathcal{B}}, s) = \{(\chi, \mathcal{L}) \in \aleph \times \aleph, | m_{\mathcal{A}} \times m_{\mathcal{B}}(\chi, \mathcal{L}) \preccurlyeq s\}$ is called an upper s-level $U(m_{\mathcal{A}} \times m_{\mathcal{B}}, s)$

Theorem. Let $\mathcal{A} = \{(\chi, m_{\mathcal{A}}(\chi))|\chi \in \aleph\}$ and $\mathcal{B} = \{(\chi, m_{\mathcal{B}}(\chi))\,|\,\chi \in \aleph\}$ be is semiprime fuzzy Bd-ideals of $\aleph$, then $\mathcal{A} \times \mathcal{B}$ is semiprime fuzzy Bd-ideals of $\aleph \times \aleph$.

Theorem. Let $\mathcal{A} = \{(\chi, m_{\mathcal{A}}(\chi)|\chi \in \aleph\}$ and $\mathcal{B} = \{(\chi, m_{\mathcal{A}}(\chi))\,|\,\chi \in \aleph\}$ be prime fuzzy Bd-ideals of $\aleph$, then the nonempty set upper s-level cut $U(m_{\mathcal{A}} \times m_{\mathcal{B}}, s)$ is prime fuzzy Bd-ideals of $\aleph \times \aleph$.

Proof. Let A and B be prime fuzzy Bd-ideals of X, therefore for any $(\chi, \mathcal{L}) \in \aleph \times \aleph$,

Let $(\chi_1, \chi_2), (\mathcal{L}_1, \mathcal{L}_2) \in \aleph \times \aleph$, such that

$(\chi_1, \chi_2) * (\mathcal{L}_1, \mathcal{L}_2) \in U(m_{\mathcal{A}} \times m_{\mathcal{B}}, s)$, then

$(\tilde{\mu}_{\mathcal{A}} \times \tilde{\mu}_{\mathcal{B}})((\chi_1, \chi_2) * (\mathcal{L}_1, \mathcal{L}_2)) = (m_{\mathcal{A}} \times m_{\mathcal{B}})(\chi_1 * \mathcal{L}_1, \chi_2 * \mathcal{L}_2)$

$\preccurlyeq$ r max $\{(m_{\mathcal{A}} \times m_{\mathcal{B}})(\chi_1, \chi_2), (m_{\mathcal{A}} \times m_{\mathcal{B}})(\mathcal{L}_1, \mathcal{L}_1)\}$

$\preccurlyeq \max\{s, s\} = s \preccurlyeq \max\{s, s\} = s$. Hence $U(m_{\mathcal{A}} \times m_{\mathcal{B}}, s)$ is prime fuzzy Bd-ideals of $\aleph \times \aleph$

In a similar way, we can prove that $(m_{\mathcal{A}} \times m_{\mathcal{B}}, s)$ is semiprime fuzzy Bd-ideals of $\aleph \times \aleph$

Theorem. Let $\mathcal{A} = \{(\chi, m_{\mathcal{A}}(\chi)|\chi \in \aleph\}$ and $\mathcal{B} = \{(\chi, m_{\mathcal{B}}(\chi))\,|\,\chi \in \aleph\}$ be semiprime fuzzy Bd-ideals of $\aleph$, then the nonempty set upper s-level cut $U(m_{\mathcal{A}} \times m_{\mathcal{B}}, s)$ is semiprime fuzzy.

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Bd-ideals of $\aleph \times \aleph$ direct product of finite prime and semiprime fuzzy Bd-ideals of Bd-algebra

Definition. Let $\mathcal{A} = (m_{\mathcal{A}_i})$ prime fuzzy set of a Bd-algebra $\aleph_i$, respectively i=1,2,…,c Then $\prod_{i=1}^{c} \mathcal{A}_i$ is called direct product of finite prime fuzzy set of a Bd-algebra of $\prod_{i=1}^{c} \aleph_i$.

$\prod_{i=1}^{c} m_{\mathcal{A}_i}\left((x_1, x_2, \ldots, x_c) * (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_c)\right) \preccurlyeq r \max\left\{\prod_{i=1}^{n} m_{\mathcal{A}_i}(x_1, x_2, \ldots, x_c), \prod_{i=1}^{c} m_{\mathcal{A}_i}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_c)\right\}$, for all $(x_1, x_2, \ldots, x_c)$,

$(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_c) \in \prod_{i=1}^{c} \aleph_i$. And is called direct product of finite semiprime fuzzy set of a Bd-algebra of $\prod_{i=1}^{c} \aleph_i$

$\prod_{i=1}^{c} m_{\mathcal{A}_i}\left((x_1, x_2, \ldots, x_c) * (x_1, x_2, \ldots, x_c)\right) \preccurlyeq r \max\left\{\prod_{i=1}^{c} m_{\mathcal{A}_i}(x_1, x_2, \ldots, x_c)\right\}$, for all $(x_1, x_2, \ldots, x_c) \in \prod_{i=1}^{c} \aleph_i$.

Example. Let $\aleph_1 = \{0, y, d\}$ and $\aleph_2 = \{0, y, d, e\}$ are Bd-algebras by the following tables:

*	0	y	e
0	0	y	y
y	y	y	y
d	d	y	d

*	0	y	d	e
0	0	y	y	y
y	y	y	y	y
d	d	d	d	d
e	e	y	y	e

Then $\aleph_1 \times \aleph_2 = \{(0,0),(0,y),(0,d),(0,e),(y,0),(y,y),(y,d),(y,e),(d,0),(d,y),(d,d),(d,e)\}$

is an Bd-algebra. We defined prime fuzzy Bd-ideals

$\mathcal{A}_1 = (m_{\mathcal{A}_1})$ on $\aleph_1$ as $m_{\mathcal{A}_1}: \aleph_1 \longrightarrow [0,1]$ by and $m(0) = 0.8$, $m(y) = 0.3$, $m(d) = 0.4$

We also introduced the concept of prime fuzzy Bd-ideals.

$\mathcal{A}_2 = (m_{\mathcal{A}_2})$ on $\aleph_2$ as $m_{\mathcal{A}_2}: \aleph_2 \longrightarrow [0,1]$ by and $m(0) = 0.9$, $m(y) = 0.7$, $m(d) = m(e) = 0.5$,

By a standard calculation $\mathcal{A}_1 \times \mathcal{A}_2 = < m_{\mathcal{A}_1 \times \mathcal{A}_2} >$ is prime fuzzy Bd-ideals of $\prod_{i=1}^{c} \aleph_i$

We also introduced the notion of semiprime fuzzy Bd-ideals.

$\mathcal{A}_1 = (p_{\mathcal{A}_i})$ on $\aleph_1$ as $p_{\mathcal{A}_1}: \aleph_1 \longrightarrow [0,1]$

$p(0) = 0.8$, $p(y) = 0.3$, $p(d) = 0.4$

Defined semiprime fuzzy Bd-ideals $\mathcal{A}_2 = (p_{\mathcal{A}_2})$ on X_2 as $p_{\mathcal{A}_2}: X_2 \longrightarrow [0,1]$ by

$p(0) = 0.8$, $p(y) = 0.3$, $p(d) = 0.4$, $p(e) = 0.7$,

By routine calculation $\mathcal{A}_1 \times \mathcal{A}_2 = < m_{\mathcal{A}_1 \times \mathcal{A}_2} >$ is semiprime fuzzy Bd-ideals of $\prod_{i=1}^{c} \aleph_i$

Theorem. Let $\mathcal{A}_i = (m_{\mathcal{A}_i})$ be prime fuzzy Bd-ideals of Bd-algebra $\aleph_i$, respectively i=1,2,3,…,c, $\prod_{i=1}^{c} \mathcal{A}_i$. is prime fuzzy Bd-ideals of $\prod_{i=1}^{c} \aleph_i$.

Proof. Let $(x_1, \ldots, x_c), (\mathcal{L}_1, \ldots, \mathcal{L}_c) \in \prod_{i=1}^{c} \aleph_i$. Then

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$\prod_{i=1}^{c} m_{\mathcal{A}_i}\{(x_1,\ldots,x_c)*(\mathcal{L}_1,\ldots,\mathcal{L}_c)\} = \{m_{\mathcal{A}_1}(x_1*\mathcal{L}_1),\ldots,m_{\mathcal{A}_2}(x_c*\mathcal{L}_c)\}$

$\preccurlyeq$ r max $\{(m_{\mathcal{A}_1}(x_1), m_{\mathcal{A}_1}(\mathcal{L}_1)),\ldots,(m_{\mathcal{A}_c}(x_c), m_{\mathcal{A}_c}(\mathcal{L}_c))\}$
=r max $\{\prod_{i=1}^{c} m_{\mathcal{A}_i}(x_1,\ldots,x_c), \prod_{i=1}^{c} m_{\mathcal{A}_i}(\mathcal{L}_1,\ldots,\mathcal{L}_c)\}$. ■

Proposition. Let $\mathcal{A}_i = (m_{\mathcal{A}_i})$ be semiprime fuzzy set of a Bd-algebra $\aleph_i$, respectively i=1,2,3,…,c. $\prod_{i=1}^{c} \mathcal{A}_i$. Is semiprime fuzzy Bd-ideals of $\prod_{i=1}^{c} \aleph_i$.

Proof. Let $(x_1,\ldots,x_c) \in \prod_{i=1}^{c} \aleph_i$. Then $\prod_{i=1}^{c} m_{\mathcal{A}_i}\{(x_1,\ldots,x_c)*(x_1,\ldots,x_c)\} = \{m_{\mathcal{A}_1}(x_1*x_1),\ldots,m_{\mathcal{A}_c}(x_c*x_c)\} \preccurlyeq$ r $\{m_{\mathcal{A}_1}(x_1),\ldots,m_{\mathcal{A}_c}(x_c)\}$.

DISCUSSION

The results of this study suggest that prime and semiprime fuzzy ideals in Bd-algebras do not lose their main features when they are combined using Cartesian or direct product, as long as the needed conditions are met. In most cases, the product structures still reflect the behavior of the original ideals in a clear way.

This shows that there is a close connection between the starting ideals and the structures formed from them, where the main properties are passed on in a natural manner.

CONCLUSION

This section studies the Cartesian and direct products of finite prime fuzzy Bd-ideals in Bd-algebras and examines the conditions under which primeness is preserved.

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