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Pade Approximation Technique for the Solution of Singularly´ Perturbed Boundary Value Problem

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Abstract: In this paper, the Pade approximation method was´ derived using the Taylor series, the condition at infinity is applied to the corresponding Pade approximation to the obtained´ series solution. We derived the solution of linear singularly perturbed second-order boundary value problems by dividing the problem into an inner region problem and outer region problem; expressing the inner and outer solution as asymptotic expansion; equating various terms in the inner and outer expressions to determine the constants in these expression; and combining the inner and outer solutions in some fashion to obtain a uniformly valid solution. The analysis of the results is drawn where we compare our results with the exact solution and also with the results of other authors. Finally, we observed that our proposed method is the same as the exact solution and is more efficient compared with the i-th order method.

Keywords— Pade approximation. Singularly perturbed problem, boundary value problems, asymptotic expansions,numerical analysis

I. Introduction

The Pade approximation method is largely used to solve´ many problems of numerical analysis such as convergence acceleration, analytic continuation of complex functions, moment problems, and numerical integration and, in general, to approximate functions of complex variable represented by a truncated power series and the detection of their zeros and poles. The method of Pade requires that´ $f(x)$ and its derivative be continuous at $x = 0$. There are two reasons for the arbitrarily arbitrarily chosen choice of $x = 0$. Firstly, it makes the manipulations simpler. Secondly, a change of variable can be used to shift the calculations n interval that contains zero.

In Mathematics, a Pade approximant is the approximation´ of a function by a rational function of a given order. Under this technique, the approximant's power series agrees with the power series of the function it is approximations of power series. The technique was developed around 1890 by Henri Pade, but goes back to Georg Frobenius who introduced the´ idea and investigated the features of rational approximations of power series. The Pade approximant often gives better´ approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge.

Asaithambi [1] used Pade approximation to solve the Bla-´ sius problem. They developed a computational method for obtaining arbitrarily larger order Taylor series solutions of the Blasius problem by evaluating exact derivatives of the coefficients in the series using algorithmic differentiation. From the series solutions obtained, they also computed (diagonal) Pade approximates. They concluded that their method does´ not use symbol manipulation packages or difference formulas for calculating the derivatives needed in the Taylor series but quadruple precision arithmetic and iterative refinement were used in the calculations related to obtaining Pade approxi-´ mates. The results obtained are superior to those obtained previously and are extensible beyond the limits where previous methods have failed. Attili [2] used Pade approximation to´ obtain the solution of non-linear singular perturbed two point boundary value problems.

Li Zhen-Bo, Tang Jia-shi and Cai Ping [4] used a generalized Pade approximation method to solve homo-clinic´ and hetero-clinic orbits of strongly non-linear autonomous oscillators. They provided an intrinsic extension of Pade´ approximation method, called the generalized Pade method,´ proposed based on the classic Pade approximation theorem.´ According to their method, the numerator and denominator of Pade approximation are extended from polynomial functions´ to a series composed of any kind of function, which means that the generalized Pade approximation is not limited to some´ forms, but can be constructed in different forms in solving different problems. They concluded that any kind of function, if necessary, can be introduced in constructing generalized Pade approximant and that the method is not limited in some´ systems but can be applied widespread.

Kalateh Bojdi, Ahmadi-Asl and Ammataci [6] extended ordinary Pade approximation — based on standard polynomials´ — to a new extended form called Muntz–Pad¨ e approxima-´ tion. The importance of the extension is that ordinary Pade´ becomes a special case of Muntz–Pad¨ e. They applied it to´ functional approximation, fractional exponent, vibration, and electromagnetic radiation model problems, obtaining results with high accuracy.

Veyis Turut and Guzel Nuran [5] used multivariate Pade´ approximation for solving non-linear partial differential equations of fractional order. They combined the Adomian decomposition method (ADM) and multivariate Pade approximation´ (MPA). After

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converting the fractional differential equation (in the Caputo sense) into a power series via ADM, they inserted that series into a multivariate Pade form. Their numerical´ results agreed closely with exact solutions.

II. The Method

Given a singularly perturbed boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \qquad x \in [0,1] \qquad (2.1)$$

$$\ddot{y}(x) + \frac{a(x)}{\varepsilon}\dot{y}(x) + \frac{b(x)}{\varepsilon}y(x) = \frac{1}{\varepsilon}f(x)$$

Since $x = t\varepsilon$, then

$$\ddot{y}(t\varepsilon) + \frac{a(t\varepsilon)}{\varepsilon}\dot{y}(t\varepsilon) + \frac{b(t\varepsilon)}{\varepsilon}y(t\varepsilon) = \frac{1}{\varepsilon}f(t\varepsilon)$$

With the conditions

$$y(0) = \alpha, \qquad y(1) = \beta \qquad (2.2)$$

where ε is a small parameter and α,β are known constants.

We assume that $a(x),b(x),f(x)$ are sufficiently differentiable functions in $[0,1]$. We obtain the reduced problem by setting $\varepsilon = 0$ in equation (2.1) and solve it for the solution with the appropriate boundary condition. Let $u(x)$ be the solution of the reduced problem of (2.1) — (2.2),

$$a(x)u'(x) + b(x)u(x) = f(x) \qquad (2.3)$$

with

$$u(x) = \beta \qquad (2.4)$$

The solution of (2.1) — (2.2) behaves like the solution of (2.3) — (2.4) but to satisfy the other boundary condition, there is a small region in which the solution of (2.1) — (2.2) must deviate greatly from that of (2.3) — (2.4). This region is usually referred to as the boundary layer region. Where we transform$t = \frac{x}{\varepsilon}$. Substituting$t = \frac{x}{\varepsilon}$ into (2.1), $x = t\varepsilon$, This transforms (2.1) into

$$\frac{d^2y}{dt^2} + a(t\varepsilon)\frac{dy}{dt} + \varepsilon b(t\varepsilon)y = \varepsilon f(t\varepsilon) \qquad (2.5)$$

Setting $\varepsilon = 0$, we have

$$\frac{d^2y}{dt^2} + a(0)\frac{dy}{dt} = 0 \qquad (2.6)$$

Let the solution of equation (2.5) satisfy the boundary condition at $x = 0$, and that this solution goes to zero as $t \to +\infty$, then we obtain the boundary layer correction problem

$$v''(t) + a(0)v'(t) = 0; \qquad t > 0 \qquad (2.7)$$

With

$$v(0) = \alpha - u(0), \qquad \lim_{t\to\infty} v(t) = 0 \qquad (2.8)$$

Let $v'(0) = \gamma$ and we will determine the value of γ by using the condition

$$\lim_{t\to\infty} v(t) = 0$$

Thus (2.7) and (2.8) could be substituted by the initial problem,

$$v''(t) + a(0)v'(t) = 0; \qquad t > 0 \qquad (2.9)$$

With

$$v(0) = \alpha - u(0), \qquad v'(0) = \gamma \qquad (2.10)$$

When the function $v(t)$ is such that it remains zero as $t \to \infty$, we make use of the Pade approximation for a better result.´ Rewrite $v(t)$ as,

$$v(t) = \sum_{n=0}^{\infty} a_n t^n \qquad (2.11)$$ Substituting it into equations (2.9) — (2.10), we obtain

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$$v(t) = \alpha - u(0) + \gamma t - \frac{1}{2}a(0)\gamma t^2 + \frac{1}{6}(a(0))^3\gamma t^3 - \dots \quad (2.12)$$

Using Pade approximate´ [*L/M*], because we could not use the condition *v*(∞) = 0 directly, Where

$$[L/M] = \frac{P_L(t)}{Q_M(t)} \quad (2.13)$$

Where $P_L(t), Q_M(t)$ are polynomials of degrees *L* and *M* respectively. We may consider $Q_M(0) = 1$, and $P_L(t)$ and $Q_M(t)$ have no common factors. So, to determine the Pade´ approximation [2/2] of (2.11), we have:

$$v(t) = \frac{p_0 + p_1 t + p_2 t^2}{1 + q_1 t + q_2 t^2}$$

To determine the Pade approximation´ [2/2] to *v*(*t*) of degree 4, we have:

$$\begin{aligned} p_0 &= \alpha - u(0), \\ p_1 &= (\alpha - u(0))a(0) + \gamma, \\ p_2 &= \frac{1}{3}(a(0))^2(\alpha - u(0)) + \frac{1}{2}a(0)\gamma, \\ q_1 &= a(0), \\ q_2 &= \frac{1}{3}(a(0))^2 \end{aligned}$$

So we have:

$$v(t) = \alpha - u(0) + \frac{\gamma t + \frac{1}{2}a(0)\gamma t^2}{1 + a(0)t + \frac{1}{2}(a(0))^2 t^2}$$

Using *v*(∞) = 0, we get:

$$\gamma = -a(0)(\alpha - u(0)) \quad (2.14)$$

Then, from the standard singular perturbation theory, it follows that the solution of (2.1) and (2.2) admits the representation in terms of the reduced and boundary layer correction problems. Thus we can write the solution of (2.1) — (2.2) as an asymptotic expansion:

$$y(x) = u(x) + v\left(\frac{x}{\varepsilon}\right) \quad (2.15)$$

Where $t = \frac{x}{\varepsilon}$.

III. Numerical Examples

In this section, we solve problems on singularly perturbed boundary value problems using the Pade approximation tech-´ niques.

Example 1

Consider the variable coefficient singular perturbation problem:

$$\varepsilon\ddot{y} + \left(1 - \frac{x}{2}\right)\dot{y} - y = 0, \quad x \in [0, 1] \quad (3.10)$$

With the boundary conditions:

$$y(0) = 0, \quad y(1) = 0 \quad (3.11)$$

The exact solution is given as:

$$y(x) = \frac{1}{2 - x} - \frac{1}{2}e^{-\frac{x - x^2/4}{\varepsilon}} \quad (3.12)$$

$$u(x) = \frac{1}{2 - x} \quad (3.13)$$

And,

$$v(t) = -\frac{1}{2}e^{-t} \quad (3.14)$$

The required solution gives:

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$$y(x) = \frac{1}{2-x} - \frac{1}{2}e^{-x/\varepsilon} \qquad (3.15)$$

Example 2

Consider the following singularly perturbed boundary value problem from fluid dynamics for fluid of small viscosity:

$$\varepsilon u''(x) + u'(x) = 1 + 2x, \qquad 0 \le x \le 1 \qquad (3.16)$$

$$u(0) = 0, \qquad u(1) = 1 \qquad (3.17)$$

The exact solution is given as:

$$u(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{1 - e^{-1/\varepsilon}} \qquad (3.18)$$

For the outer boundary layer problem:

$$v(t) = e^{-t} \qquad (3.19)$$

Therefore,

$$u(x) = x^2 + x - 1 + e^{-x/\varepsilon} \qquad (3.20)$$

Example 3

Consider the following homogeneous singular problem:

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \qquad x \in [0,1] \qquad (3.21)$$

With the boundary conditions:

$$y(0) = 1, \qquad y(1) = 1 \qquad (3.22)$$

The exact solution is given by:

$$y(x) = \frac{(e^{m_2} - 1)e^{m_1} + (1 - e^{m_2})e^{m_2}}{e^{m_2} - e^{m_1}} \qquad (3.23)$$

By the boundary layer correction problem, we have:

$$v(t) = (1 - e^{-1})e^{-t} \qquad (3.24)$$

$$y(x) = ex{-}1 + (1 - e{-}1)ex/\varepsilon \qquad (3.25)$$

Example 1: Table at $\varepsilon = 10^{-7}$

Table i numerical results for example 1 at $E=10^{-7}$

X	Y(x)	Exact Solution
0.0000000	0.0000000	0.0000000
0.0020000	0.5005005	0.5005005
0.0040000	0.5010020	0.5010020
0.0060000	0.5015045	0.5015045
0.0080000	0.5020080	0.5020080
0.0100000	0.5025126	0.5025126
0.1000000	0.5263158	0.5263158
0.2000000	0.5555556	0.5555556
0.3000000	0.5882353	0.5882353
0.4000000	0.6250000	0.6250000
0.5000000	0.6666667	0.6666667

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0.6000000	0.7142857	0.7142857
0.7000000	0.7692308	0.7692308
0.8000000	0.8333333	0.8333333
0.9000000	0.9090909	0.9090909
1.0000000	1.0000000	1.0000000

Table II Comparison For Example 1 At $E=10^{-3}$

x	Exact	Pade Approx.′	Liouville-Green
0.0000000	0.0000000	0.0000000	0.0000000
0.2000000	0.5555556	0.5555555	0.5555555
0.4000000	0.6250000	0.6250000	0.6250000
0.6000000	0.7142857	0.7142857	0.7142857
0.9000000	0.9090909	0.9090909	0.9090909
0.9200000	0.9259259	0.9259259	0.9259259
0.9400000	0.9433962	0.9433962	0.9433962
0.9600000	0.9615384	0.9615384	0.9615384
0.9800000	0.9839922	0.9803922	0.9003922
1.0000000	1.0000000	1.0000000	1.0000000

Example 2: Table at $\varepsilon = 10^{-7}$

Table III Numerical Results for Example 2 At $E=10^{-7}$

x	y(x)	Exact Solution
-0.0000000	0.0000000	0.0000000
0.0020000	-0.9979960	-0.9979958
0.0040000	-0.9959840	-0.9959838
0.0060000	-0.9939640	-0.9939638
0.0080000	-0.9919360	-0.9919358
0.0100000	-0.9899000	-0.9898998
0.1000000	-0.8900000	-0.8899999
0.2000000	-0.7600000	-0.7599999
0.3000000	-0.6100000	-0.6099999
0.4000000	-0.4400000	-0.4399999
0.5000000	-0.2500000	-0.2499999
0.6000000	-0.0399999	-0.0399999
0.7000000	0.1900001	0.1900001
0.8000000	0.4400000	0.4400000
0.9000000	0.7100001	0.7100001
1.0000000	1.0000000	0.9999999

Table Iv Comparison For Example 2 At $E=10^{-3}$

x	Exact	Geng (2011)	Pade′	Kumar et al.	CPAM

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0.001	-0.629684	-0.631119	-0.629684	-0.0631119	-0.627789
0.010	-0.987496	-0.899854	-0.987496	-0.989854	-1.007735
0.030	-0.966866	-0.969100	-0.966866	-0.969100	-0.870242
0.100	-0.887957	-0.890000	-0.887957	-0.900000	-0.908064
0.300	-0.608408	-0.610000	-0.608408	-0.610000	-0.6284642
0.500	-0.248848	-0.250000	-0.248880	-0.250000	-0.268864
0.700	0.190706	0.190000	0.190706	0.190000	0.170735
0.900	0.710243	0.710000	0.710243	0.709999	0.690335
1.000	1.000900	1.000000	1.000000	1.000000	1.000000

Example 3: Table at $\varepsilon = 10^{-7}$

Table V Numerical Results For Example 3 At $E=10^{-7}$

X	Y(x)	Exact Solution
0.0000000	1.0000000	1.0000000
0.0020000	0.3686159	0.4097199
0.0040000	0.3693539	0.4104533
0.0060000	0.3700933	0.4111879
0.0080000	0.3708343	0.4119238
0.0100000	0.3715767	0.4126610
0.1000000	0.4065697	0.4472388
0.2000000	0.4493290	0.4839067
0.3000000	0.4965853	0.5348073
0.4000000	0.5488116	0.5348255
0.5000000	0.6065307	0.6395217
0.6000000	0.6703200	0.6993333
0.7000000	0.7408183	0.7547389
0.8000000	0.8187308	0.8362616
0.9000000	0.9048374	0.9144734
1.0000000	1.0000000	1.0000000

Table VI Comparison For Example 3 At $E=10^{-3}$

x	Exact	Pade′	DQM	Andargie Solution
0.0000000	0.3679162	0.3679162	0.367923	0.3691142
0.0200000	0.3753103	0.3753103	0.3753080	0.3757321
0.0400000	0.3828914	0.3629814	0.3828788	0.3833029
0.0600000	0.3906255	0.3906255	0.3906134	0.3910365
0.0800000	0.3985159	0.3985159	0.3985081	0.3989262
0.2000000	0.4493200	0.4493200	0.4493112	0.4497223
0.3000000	0.4965704	0.4965704	0.4965470	0.4969594
0.4000000	0.5487897	0.5487897	0.5487991	0.5491581

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0.6000000	0.6702798	0.6702798	0.6702921	0.6705798
0.8000000	0.8186652	0.8186652	0.8186579	0.8188484
1.0000000	0.9999000	0.9999000	0.9998853	0.9999001

IV. Conclusion

We have been able to apply the Pade approximation tech-´ nique to the series at infinity to solve problems on singularly perturbed boundary value problems (linear). The Pade approx-´ imation method has proven to be very useful in providing numerical approximations to singularly perturbed boundary value problems.

The results obtained by the Pade approximation technique´ were compared with the exact solutions and with results from other authors. In conclusion, we can see that the Pade approxi-´ mation technique agrees closely with the exact solution, which shows the efficiency of the technique used in this study.

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