

Volterra Integral Equations: A Numerical Solution Method Using Shifted Chebyshev Polynomial

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Abstract: This study presents a numerical method for solving Volterra integral equations of the second kind using shifted Chebyshev polynomials. Volterra integral equations arise in various scientific and engineering applications, including population dynamics, physics, and control systems. Due to their complexity, obtaining analytical solutions is often challenging, making numerical techniques crucial. We employ shifted Chebyshev polynomials as basis functions to approximate the solution, transforming the integral equation into a system of algebraic equations. The shifted Chebyshev polynomials offer excellent approximation properties, improving convergence rates and accuracy. The proposed method is analyzed for stability and efficiency, and numerical experiments demonstrate its effectiveness in solving different classes of Volterra integral equations. The results highlight the advantages of the approach compared to traditional numerical methods.

Keywords: Volterra, Integral equations, Approximate solution, Shifted Chebyshev Polynomial.

Mathematics Subject Classification: 65C30, 65L06, 65C03

I. Introduction

Integral equations are equations in which an unknown function appears under an integral sign. It is applicable in several fields, such as engineering and science. Various classification methods for integral equations exist. A few standard classifications include distinctions between linear and nonlinear, homogenous and non-homogenous, Fredholm and Volterra, first order, second order, and third order, and singular and regular integral equations. These distinctions usually rest on some fundamental property, such as the consideration of the linearity of the equation or the homogeneity of the equation. Integral equations are contained in many mathematical formulations of physical phenomena; as a result, there is a need for numerical methods of solving such equations. (Zarnan, 2018).

The Volterra integral equation's numerical solutions were investigated and taken into consideration by numerous academics, including: Bernstein method (Adhraa and Ayal, 2019), Adomian decompositions method (Khan and Bakoda, 2013), Collocation method by (Ajileye and Amoo, 2023; Agbolade and Anake, 2017), Hybrid linear multistep method (Mehdiyera, Ibrahim and Imanova, 2019), Chebyshev-Galerkin method (Issa and Saleh, 2017), Lagrange Interpolation (Shoukralla and Ahmed, 2020), Least-Squares Method (Al-Humedi and Shoushan, 2021), Chebyshev polynomials (Maadadi & Rahmoune, 2018), Optimal Auxiliary Function Method (OAFM) (Zada, Al-Hamami, Nawaz, Jehanzeb, Morsy, Abdei-Aty and Nisar, 2021) and many more.

This paper considered Volterra integral equation of the second kind of the form

$$y(x) + \lambda \int_0^x k(x, t) y(t) dt = f(x) \quad (1)$$

where $k(x, t)$ is the Volterra integral kernel, λ is the parameter given, $f(x)$ is a known function, and $y(x)$ is the unknown function to be determined.

Basic Terms and Definitions

We give certain definitions and fundamental notions in this section for the purpose of problem formulation.

Definition 1 :(Agbolade, & Anake, 2017) The desired collocation points within an interval are determined using this method. i.e., $[a, b]$, and is provided by

$$l_u = a + \frac{(b-a)u}{M}, u = 0, 1, 2 \dots \dots M, a \leq x \leq b$$

Definition 2: Shifted Chebyshev Polynomials (Zhang, M., 2008)

This is a special type of polynomial generated from Chebyshev polynomial. The shifted Chebyshev has interval $[0,1]$, while Chebyshev has an interval of $[-1,1]$.

The shifted Chebyshev is generated using;

$$T_n^*(x) = T_n(2x - 1), n \geq 0$$

Where $T_n^*(x)$ is the shifted Chebyshev term, while $T_n(x)$ is the Chebyshev term.

Few terms of the shifted Chebyshev terms are;

$$T_0^*(x) = 1$$

$$T_1^*(x) = 2x - 1$$

$$T_2^*(x) = 8x^2 - 8x + 1$$

$$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$$

$$T_4^*(x) = 128x^4 - 256x^3 + 160x^2 + 32x + 1$$

II. Materials and Methods

The approximation method for the numerical solution of Volterra integral equations is implemented in this section.

Method of Solution

Let the solution of equation (1) be approximated by

$$y(x) = \sum_{i=0}^M a_i L_i(x) = L(x)A \quad (2)$$

where

$$L_i(x) = \sum_{k=0}^N \left((-1)^i 2^{2k-2i} \frac{\Gamma(2k-i+2)}{\Gamma(i+1)\Gamma(2k-2i+2)} x^{k-i} \right), k = 0, \dots, N \text{ and}$$

$$A = [a_0 \ a_1 \ \dots \ a_M]^T$$

Substituting equation (2) into equation (1) gives

$$L(x)A + \lambda \int_0^x k(x, t) L(t) A dt = f(x) \quad (3)$$

Collocating equation (3) at x_i using standard collocation points equation

$$x_i = a + \frac{(b-a)i}{N}, i = 0, 1, 2, \dots, N$$

$$L(x_i)A + \lambda \int_0^{x_i} k(x_i, t) L(t) A dt = f(x_i) \quad (4)$$

Factorizing the value of A from equation (4) gives

$$[L(x_i) + \lambda \int_0^{x_i} k(x_i, t) L(t) dt] A = f(x_i) \quad (5)$$

Equation (5) can be in the form

$$G(x_i)A = f(x_i) \quad (6)$$

where

$$G(x_i) = L(x_i) + \lambda \int_0^{x_i} k(x_i, t) L(t) dt$$

$$G(x_i) = \begin{bmatrix} G_0(x_0) & G_1(x_0) & G_2(x_0) & \dots & G_N(x_0) \\ G_0(x_1) & G_1(x_1) & G_2(x_1) & \dots & G_N(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ G_0(x_N) & G_1(x_N) & G_2(x_N) & \dots & G_N(x_N) \end{bmatrix}, f(x_i) = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \quad -$$

In order to determine the value of the unknown coefficient, we used the shifted Chebyshev polynomial as the base polynomial function to solve the system of equations (6) and obtain the numerical result.

Numerical Examples

This section provides numerical examples to assess the accuracy and usefulness of the method. Let $y_n(x)$ and $y(x)$ be the approximate and exact solutions, respectively. $Error_N = |y_n(x) - y(x)|$

Example 1: Zarnan et al. (2018) Consider Volterra integral equation

$$y(x) - \int_0^x (t-x)y(t)dt = x \quad (7)$$

Exact solution: $y(x) = \sin x$

Solution 1

The approximate solution of equation (7) at N=5 gives

$$y_5(x) = 0.007252554428x^5 + 0.001612915971x^4 - 0.167616514900x^3 + 0.000244039476x^2 + 0.99997803300x + 4.662936703000 \times 10^{-14}$$

Table 1: Exact, approximate and absolute error values for Example 1

X	Exact	Our Method _{N=5}	Errors ₅	Error _{Zarnan et al.=10}
0.2	0.198669330800	0.198669339600	8.80e-9	3.31e-4
0.4	0.389418342300	0.389418363500	2.12e-8	6.42e-4
0.6	0.564642473400	0.564642505500	3.21e-8	9.14e-4
0.8	0.717356090900	0.717356131600	4.07e-8	1.13e-3
1.0	0.841470984800	0.841471038300	5.35e-8	1.28e-3

Example 2: Shoukralla et al. (2020) Consider Volterra integral equation

$$y(x) - \int_0^x (x-t)y(t)dt = x + 1 \quad (8)$$

Exact solution: $y(x) = e^x$

Solution 2

The approximate solution of equation (8) at N=5 gives

$$y_5 = 0.013854339070x^5 + 0.034866746880x^4 + 0.170409054000x^3 + 0.499046321900x^2 + 1.000082629000x + 1.000000000000$$

Table 2: Exact, approximate and absolute error values for Example 2

x	Exact	Our Method _{N=5}	Error ₅	Error _{Shoukralla et al.=5}
0.2	1.221402758000	1.221402791000	3.30e-8	9.149e-08
0.4	1.491824698000	1.491824780000	8.20e-8	6.031e-06
0.6	1.822118800000	1.822118933000	1.33e-7	7.08e-05
0.8	2.225540928000	2.225541114000	1.86e-7	4.10e-04
1.0	2.718281828000	2.718282091000	2.63e-7	1.61e-03

Example 3: Shoukralla et al. (2020) Consider Volterra integral equation

$$y(x) - \int_0^x e^x y(t)dt = \cos(x) - e^x \sin(x) \quad (9)$$

Exact solution: $y(x) = \cos(x)$

Solution 3

The approximate solution of equation (8) at N=5 gives

$$y_5 = -0.40082432e - 2x^5 + 0.461544960e - 1x^4 - 0.23786496e - 2x^3 - 0.4994146304x^2 - 0.531106e - 4x + 1.0000000000$$

Table 3: Exact, approximate and absolute error values for Example 3

X	Exact	Our Method _{N=5}	Errors	Error _{Shoukralla et al.=5}
0.2	0.9800665778	0.9800663280	2.4980e-8	9.149e-08
0.4	0.9210609940	0.9210606920	3.0200e-7	6.031e-06
0.6	0.8253356149	0.8253350202	5.9470e-7	7.08e-05
0.8	0.6967067093	0.6967057398	9.6950e-7	4.10e-04
1.0	0.5403023059	0.5402998622	24437e-6	1.61e-03

III. Results and Discussion

The numerical results derived from the solved examples using the suggested numerical approach are covered in this section.

As indicated in Table 1, the outcome for Example 1 is that the approximate solution at $N = 5$ gives $y_5 = -3.552713678801 * 10^{-14} + 0.999978043885x + 0.244040421e - 3x^2 - 0.167616536724x^3 + 0.1612963621e - 2x^4 + 0.7252526964e - 2x^5$.

The numerical results converged to an accurate solution, confirming that our method at $N=5$ outperformed the method presented by Zarnan et al. (2018) at $N=10$.

In numerical Example 2, as shown in Table 2, the approximate solution at $N = 5$ gives $y_5 = 1.000000000000 + 1.000082628954x + .499069314450x^2 + 0.170409090701x^3$

$+0.34866686678e - 1x^4 + 0.13854370802e - 1x^5$ The numerical results give a better result compared to the result obtained by Shoukralla et al. (2020) at the same value of N .

In numerical Example 3, as shown in Table 3, the approximate solution at $N = 5$ gives $y_5 = -0.40082432e - 2x^5 + 0.461544960e - 1x^4 - 0.23786496e - 2x^3 - 0.4994146304x^2 - 0.531106e - 4x + 1.0000000000$. The numerical results give a better result compared to the result obtained by Shoukralla et al. (2020) at the same value of N .

IV. Conclusions

In this work, the numerical solution of Volterra integral equations using the shifted Chebyshev as a basis function was investigated. This method is easy to compute, dependable, and efficient. All of the calculations in this work are performed using Maple 18. Taking into account a few issues demonstrates the method's reliability.

References

1. Adhraa M. M. & Ayal A. M. (2019). Numerical solution of linear Volterra integral equations with delay using Bernstein polynomial. *International Electronic Journal of Mathematics Education*, 14(3), 735-740.
2. Agbolade, A. O. & Anake, T. A. (2017). Solution of first order Volterra linear integro differential equations by collocation method. *Journal of Applied Mathematics*, 1-5. Article ID: 1510267.doi:10.1155/2017/ 15267.
3. Ajileye, G. & Amoo, S. A. (2023). Numerical solution to Volterra Integro-differential equations using collocation approximation. *Mathematics and Computational Sciences*, 4(1), 1-8.
4. Al-Humedi H. O & Shoushan A. F. (2021). Numerical solution of mixed integro-differential equations by Least-squares method and Laguerre polynomial. *Earthline of Journal mathematical Sciences*, 6(2), 309-323.
5. Issa, K. & Saleh, F. (2017). Approximate solution of Perturbed Volterra Fredholm Integro differential equation by Chebyshev-Galerkin method. *Journal of Mathematics*. doi:10,1155/2017/8213932.
6. Khan, R. H. & Bakodah, H. O. (2013). Adomian decomposition method and its modification for nonlinear Abel's integral equations. *Computers and Mathematics with Applications*. 7, 2349-2358.
7. Maadadi, A. & Rahmoune, A. (2018). Numerical solution of nonlinear Fredholm Integro-differential equations using Chebyshev polynomials. *International Journal of Advanced Scientific and technical Research*, 8(4), 85-91. <https://dx.doi.org/10.26808/rs.st.i8v4.09>.
8. Mehdiyeva, G., Ibrahimov, V. & Imanova, M. (2019). On the Construction of the Multistep Methods to Solving the Initial-Value Problem for ODE and the Volterra Integro-Differential Equations. *IAPE, Oxford, United Kingdom*, ISBN: 978-1-912532-05-6.
9. Shoukralla, E. S. & Ahmed, B. M (2020). Numerical Solution of Volterra integral equation of the second kind using Langrange interpolation via the Vandermonde matrix. *Journal of Physics: Conference Series*, 1447.

10. Zada, L. Al-Hamami, M., Nawaz, R., Jehanzeb, S., Morsy, A., Abdel-Aty, A. & Nisar, K. S.(2021).A New Approach for Solving Fredholm Integro-Differential Equations. Information Sciences Getters, 10 (3), 3-10.
11. Zarnan J. A. (2018). Numerical Solution of Volterra integral equations of second kind. International Journal of Computer Science and Mobile Computing. 5(7), 509-514.