

Multi-Shock Wave Solution of The Damped Forced Burger Equation by Homogenization of Degree

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DOI: <https://doi.org/10.51583/IJLTEMAS.2025.140500049>

Received: 30 May 2025; Accepted: 03 June 2025; Published: 10 June 2025

Abstract: This article presents an investigation into the exact solution of the damped forced Burger equation with constant coefficients, utilizing a homogenization technique. By introducing a suitable change of dependent variable, the partial differential equation is transformed into a homogeneous equation of a specific degree. Subsequently, shock waves are computed employing a perturbation-like scheme that leverages linear and nonlinear operators in a finite number of steps. The study further illustrates the behavior of 2-shock and 3-shock waves under different forcing terms, providing valuable insights into the dynamics of the system. This approach enables a deeper understanding of the complex interactions between nonlinearity, damping, and external forcing in the Burger equation.

Keywords: Burger's equation, shock waves, damping, force, homogenization of degree, etc.

I. Introduction

Nonlinear equations have a wide array of applications across various fields, describing isolated wave motions localized in small spatial regions. These applications span multiple disciplines, including hydrodynamics ([1]-[2]) plasma physics ([3], [4]) where such equations are crucial for understanding complex phenomena. Analytical solutions to nonlinear partial differential equations (PDEs) play a vital role in nonlinear sciences, particularly in nonlinear physical science, as they provide valuable physical insights and facilitate further applications. A plethora of methods has been developed over the years to seek exact solutions to nonlinear PDEs. Some of the notable methods include the (G'/G)-expansion method ([5]- [6]), the exp-function method ([7]), the sine-cosine method ([8]), the Jacobi elliptic method ([9]-[10]), the F-expansion method ([11]), the Bäcklund transformation method ([12]), the Darboux transformation method ([13]), the homogeneous balance method ([14]-[15]), the $\exp(-\phi(\eta))$ -expansion method ([16]-[17]), the tanh method ([18]). These methods have been extensively applied to study the Burger equation ([19]-[21]) demonstrating analytical and traveling wave solutions. In soliton theory, Hirota's method ([22]-[24]) is a powerful approach for obtaining soliton solutions of nonlinear integrable PDEs, which involves recasting the PDE into bilinear form. However, this requirement can be limiting. In our approach, we utilize homogenization of degree, which allows for the determination of solitary wave solutions without the need for bilinear forms or integrability constraints. This method employs a perturbation-like scheme involving linear and nonlinear operators to solve the equation directly, providing a valuable alternative for studying nonlinear PDEs.

Finding the Hirota transformation can be challenging, often requiring insight and ingenuity. Hietarinta ([25]-[26]) provides valuable guidance, based on experience, to aid in identifying suitable candidates and reducing guesswork. However, determining the appropriate bilinear form for the homogeneous equation can also be difficult, particularly for cubic or quartic equations. Such cases may require decoupling into a pair of bilinear equations, potentially involving additional independent variables or functions. To address this complexity, our approach will bypass the bilinear form. The simplified Hirota method as proven effective in finding solitary wave and soliton solutions, with notable contributions from researchers like Wazwaz. Wazwaz has extensively applied this method to derive bi-soliton ([27]-[29]) and soliton solutions for various partial differential equations (PDEs) involving one or more space variables ([30]-[31]). Additional studies have further demonstrated its applicability to PDEs with multiple space variables ([32]) showcasing the method's versatility and utility in solving complex nonlinear equations. Here we consider the non-autonomous Burger's equation with constants coefficients is given by

$$\frac{\partial u}{\partial t} + Au \frac{\partial u}{\partial x} - B \frac{\partial^2 u}{\partial x^2} + Cu + D \frac{\partial u}{\partial x} = g(t) \quad (1)$$

where $g(t)$ is the external forcing term, D is the velocity coefficient, C is the collision coefficient, B is known as the coefficient of diffusion and A is the non-linear coefficient. Here we applied the simplified the Hirota method successfully and obtained multi-shock wave solution of the damped forced Burger's equation and graphical demonstration of the solution is presented.

Multi-shock wave solution of the non-autonomous Burger's equation

Let us consider the following transformation

$$u = p(t) \frac{\partial v}{\partial x} + q(t) \quad (2)$$

Using this (2) into the equation (1) and on the simplification , we have

$$\frac{\partial^2 v}{\partial x \partial t} + Ap(t) \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} + Aq(t) \frac{\partial v}{\partial x} - B \frac{\partial^3 v}{\partial x^3} + D \frac{\partial v}{\partial x} = 0 \quad (3)$$

with the choices $p(t) = c_1 e^{-ct}$ and $q(t) = e^{ct} \int e^{-ct} g(t) dt$. Integrating equation (2) with respect to x and setting integrating constant zero, we get

$$\frac{\partial v}{\partial t} + \frac{1}{2} Ap(t) \left(\frac{\partial v}{\partial x} \right)^2 + Aq(t) \frac{\partial v}{\partial x} - B \frac{\partial^2 v}{\partial x^2} + D \frac{\partial v}{\partial x} = 0 \quad (4)$$

Now this equation (4) can be linearized by a Logarithmic transformation $v = 2 \ln(f(x, t))$. Using it in the equation (4), we get

$$\frac{1}{f} \frac{\partial f}{\partial t} + Aq(t) \frac{1}{f} \frac{\partial f}{\partial x} - B \frac{1}{f} \frac{\partial^2 f}{\partial x^2} + D \frac{1}{f} \frac{\partial f}{\partial x} + p(t) A \frac{1}{f^2} \left(\frac{\partial f}{\partial x} \right)^2 + B \frac{1}{f^2} \left(\frac{\partial f}{\partial x} \right)^2 = 0 \quad (5)$$

Decoupling it into two parts, we get

$$\frac{1}{f} \frac{\partial f}{\partial t} + Aq(t) \frac{1}{f} \frac{\partial f}{\partial x} - B \frac{1}{f} \frac{\partial^2 f}{\partial x^2} + D \frac{1}{f} \frac{\partial f}{\partial x} = 0$$

or

$$\frac{\partial f}{\partial t} + Aq(t) \frac{\partial f}{\partial x} - B \frac{\partial^2 f}{\partial x^2} + D \frac{\partial f}{\partial x} = 0$$

And

$$p(t) A \frac{1}{f^2} \left(\frac{\partial f}{\partial x} \right)^2 + B \frac{1}{f^2} \left(\frac{\partial f}{\partial x} \right)^2 = 0$$

or

$$(p(t) A + B) \left(\frac{\partial f}{\partial x} \right)^2 = 0$$

Thus this transformation provides us to rewrite the non-autonomous Burger's equation in homogeneous form. This transformation simplifies the equation and make it easier to solve. Under the assumption

$p(t) A + B = 0$, we have the equation

$$\frac{\partial f}{\partial t} + Aq(t) \frac{\partial f}{\partial x} - B \frac{\partial^2 f}{\partial x^2} + D \frac{\partial f}{\partial x} = 0 \quad (6)$$

The resulting equation is a linear partial differential equation in the variable f , which can be solved using standard techniques. Once the solution for f is obtained, we can transform it back to the original variable u to get the solution to the non-autonomous Burgers equation.

One-shock wave

Set $f = 1 + e^{kx+wt+\delta}$ in the linear equation (6). Then collect the coefficient of exponent, we get

$$w = -Aq(t)k + Bk^2 - Dk \quad (7)$$

Substitute this into the transformation (2), one can get a shock wave solution of the equation (1)

$$u = 2c_1 e^{-ct} \frac{\partial}{\partial x} \ln(f) + e^{ct} \int e^{-ct} g(t) dt$$

or

$$\begin{aligned} u &= 2c_1 e^{-ct} \frac{f_x}{f} + e^{ct} \int e^{-ct} g(t) dt \\ &= 2c_1 e^{-ct} \frac{k e^{\theta}}{1 + e^{\theta}} + e^{ct} \int e^{-ct} g(t) dt \\ &= c_1 e^{-ct} k \frac{2e^{\frac{\theta}{2}}}{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}}} + e^{ct} \int e^{-ct} g(t) dt \\ &= c_1 e^{-ct} k \frac{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}} + e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}}{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}}} + e^{ct} \int e^{-ct} g(t) dt \end{aligned}$$

$$\begin{aligned}
 &= c_1 e^{-Ct} k \left(1 + \frac{\frac{\theta}{e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}}}{\frac{\theta}{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}}}} \right) + e^{Ct} \int e^{-Ct} g(t) dt \\
 &= c_1 e^{-Ct} k \left(1 + \tanh\left(\frac{\theta}{2}\right) \right) + e^{Ct} \int e^{-Ct} g(t) dt
 \end{aligned} \quad (8)$$

Where $\theta = kx + (-Aq(t)k + Bk^2 - Dk)t + \delta$ and c_1 is arbitrary constants.

Two shock wave

Substituting $f = 1 + e^{\theta_1} + e^{\theta_2}$ in the equation (6) and equating the coefficients of exponents zero, we obtained $w_i = (-Aq(t)k_i + Bk_i^2 - Dk_i)$, $i = 1, 2$.

Then putting this in the transformation equation (2), we obtained second order shock wave solution of the equation (1) as

$$u = 2c_1 e^{-Ct} \frac{k_1 e^{\theta_1} + k_2 e^{\theta_2}}{1 + e^{\theta_1} + e^{\theta_2}} + e^{Ct} \int e^{-Ct} g(t) dt \quad (9)$$

Where $\theta_i = k_i x + (-Aq(t)k_i + Bk_i^2 - Dk_i)t + \delta_i$, $i = 1, 2$.

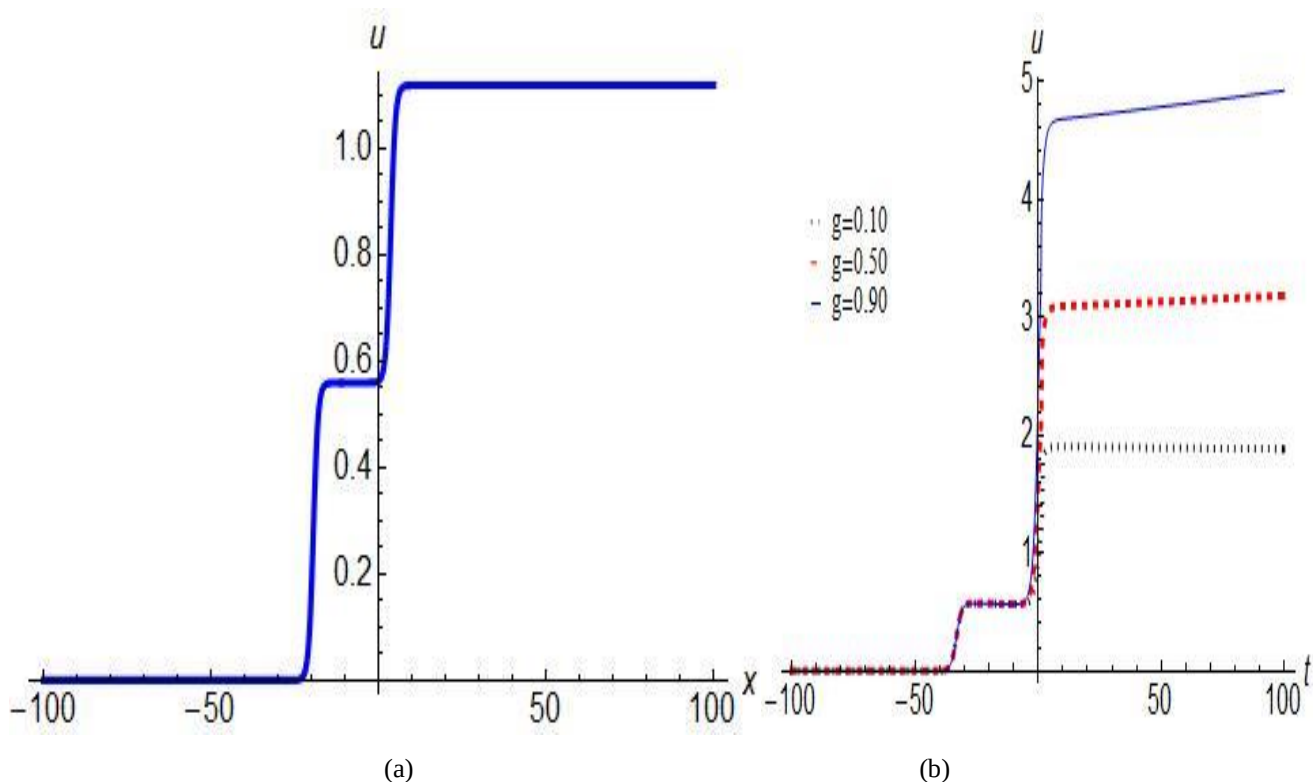
2.n-shock wave

Substituting $f = 1 + \sum_{i=1}^n e^{\theta_i}$ in the equation (6) and equating the coefficients of exponents zero, we obtained, $w_i = (-Aq(t)k_i + Bk_i^2 - Dk_i)$, $i = 1, 2, \dots, n$.

Then putting this in the transformation equation (2), we obtained n-order shock wave solution of the equation (1) as

$$u = 2c_1 e^{-Ct} \frac{\sum_{i=1}^n k_i e^{\theta_i}}{1 + \sum_{i=1}^n e^{\theta_i}} + e^{Ct} \int e^{-Ct} g(t) dt \quad (10)$$

Where $\theta_i = k_i x + (-Aq(t)k_i + Bk_i^2 - Dk_i)t + \delta_i$, $i = 1, 2, \dots, n$.



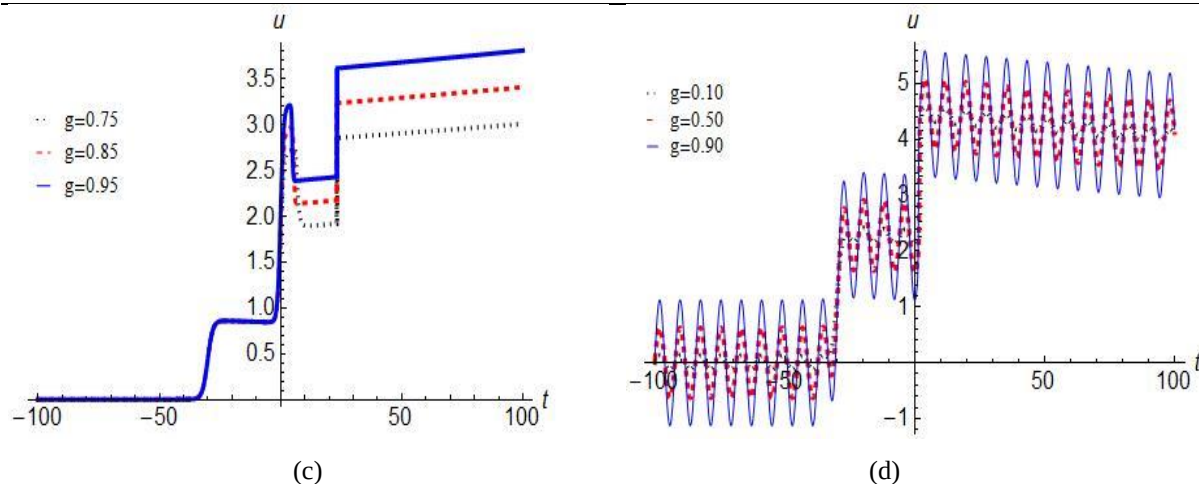


Fig. 1: 2D profile of the 2 kink soliton solution of the Eq. (1), with the following numeric choice of the parameters $c_1 = 0.2, A = 2, B = 2, C = 0.001, D = 2.5, k_1 = 1.4, k_2 = 2.8, \delta_1 = 25, \delta_2 = 2$; (a) $t = 2, f(t) = 0$; (b) $x = 4, w = 0.8, f(t) = g \operatorname{sech}(wt)$; (c) $x = 2, w = 0.8, f(t) = g \operatorname{sech}^2(wt)$; (d) $x = 2, w = 0.8, f(t) = g \sin(wt)$.

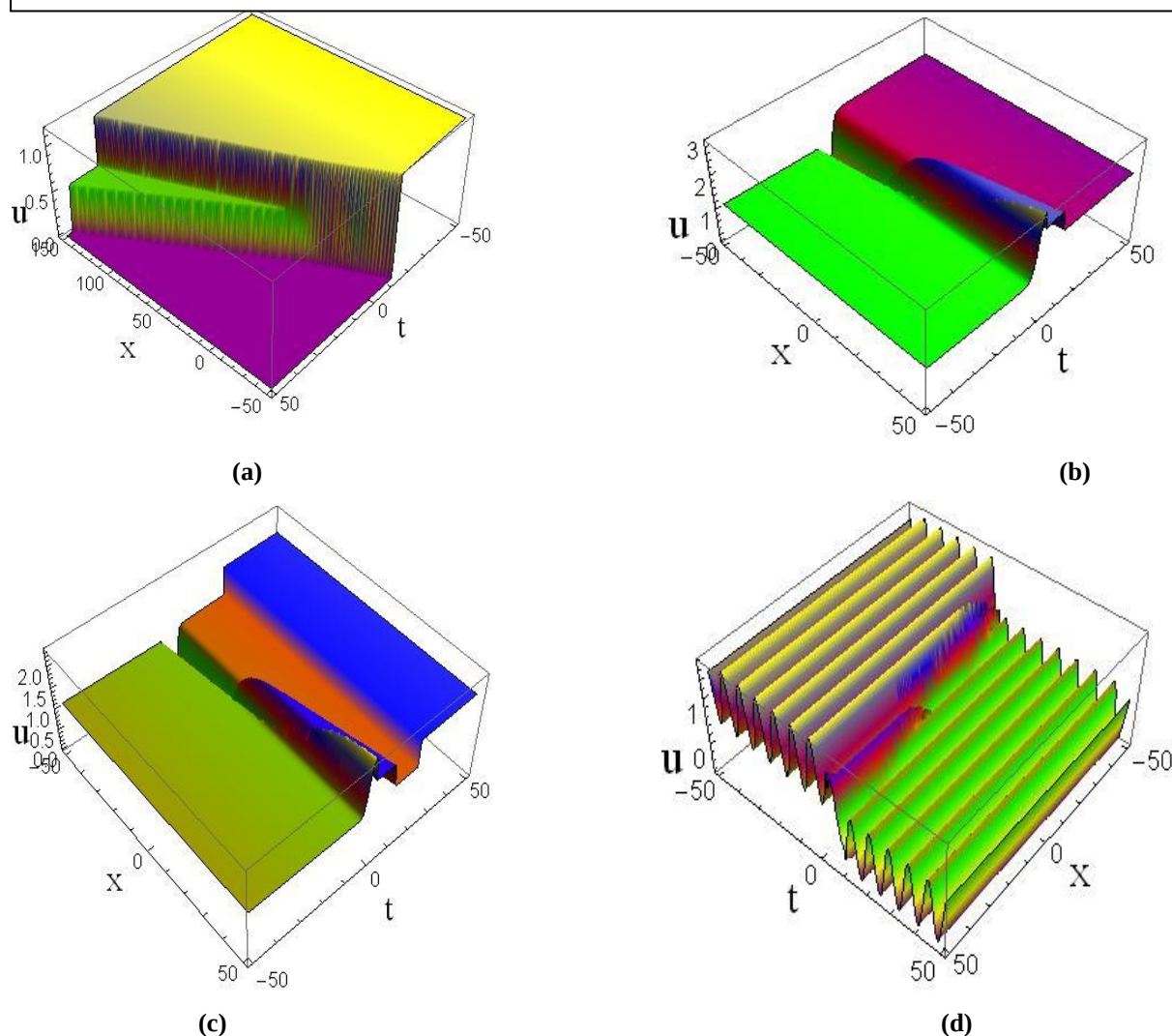


Fig. 2 3D profile of the 2 kink soliton solution of the Eq. (1), with the following numeric choice of the parameters $c_1 = 0.2, A = -2.5, B = -0.25, C = 0.001, D = 2.5, k_1 = 1.4, k_2 = 2.8, \delta_1 = 0, \delta_2 = 0$; (a) $f(t) = 0$; (b) $g = 0.5, w = 0.8, f(t) = g \operatorname{sech}(wt)$; (c) $g = 0.5, w = 0.8, f(t) = g \operatorname{sech}^2(wt)$; (d) $g = 0.5, w = 0.8, f(t) = g \sin(wt)$.

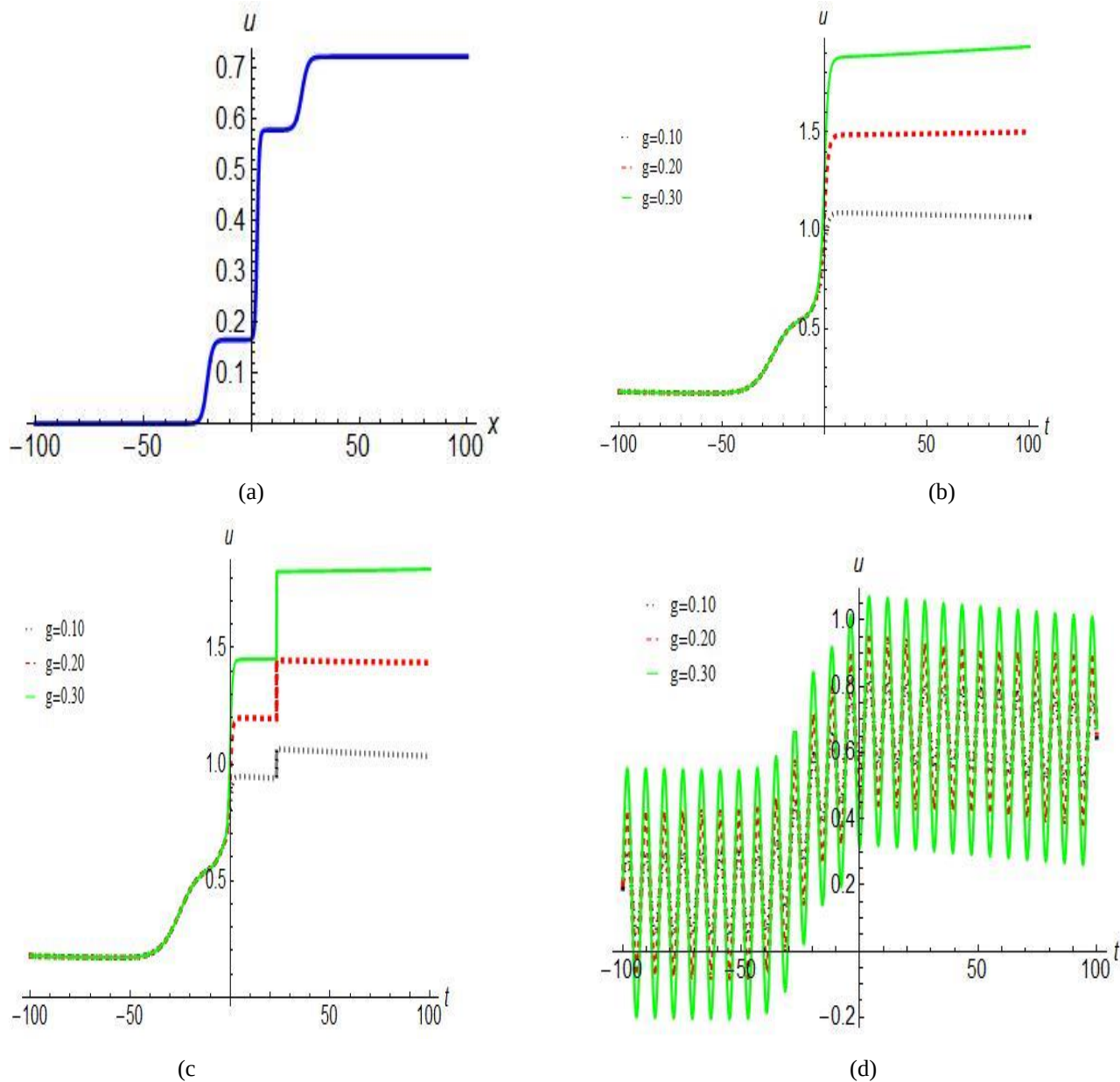


Fig. 3: 2D profile of the 3-kink soliton solution of the Eq. (1), with the numeric choice of the parameters $c_1 = 0.1, A = 1.2, B = 0.25, C = 0.001, D = 0.8, k_1 = 0.8, k_2 = 2.8, k_3 = 3.5, \delta_1 = 1, \delta_2 = 2, \delta_3 = 3$, (a) $w = 0.8, f(t) = 0, t = -32$; (b) $x = 2, w = 0.4, f(t) = g \operatorname{sech}(wt)$; (c) $x = 2, w = 0.8, f(t) = g \operatorname{sech}^2(wt)$; (d) $x = 2, w = 0.2, f(t) = g \sin(wt)$.

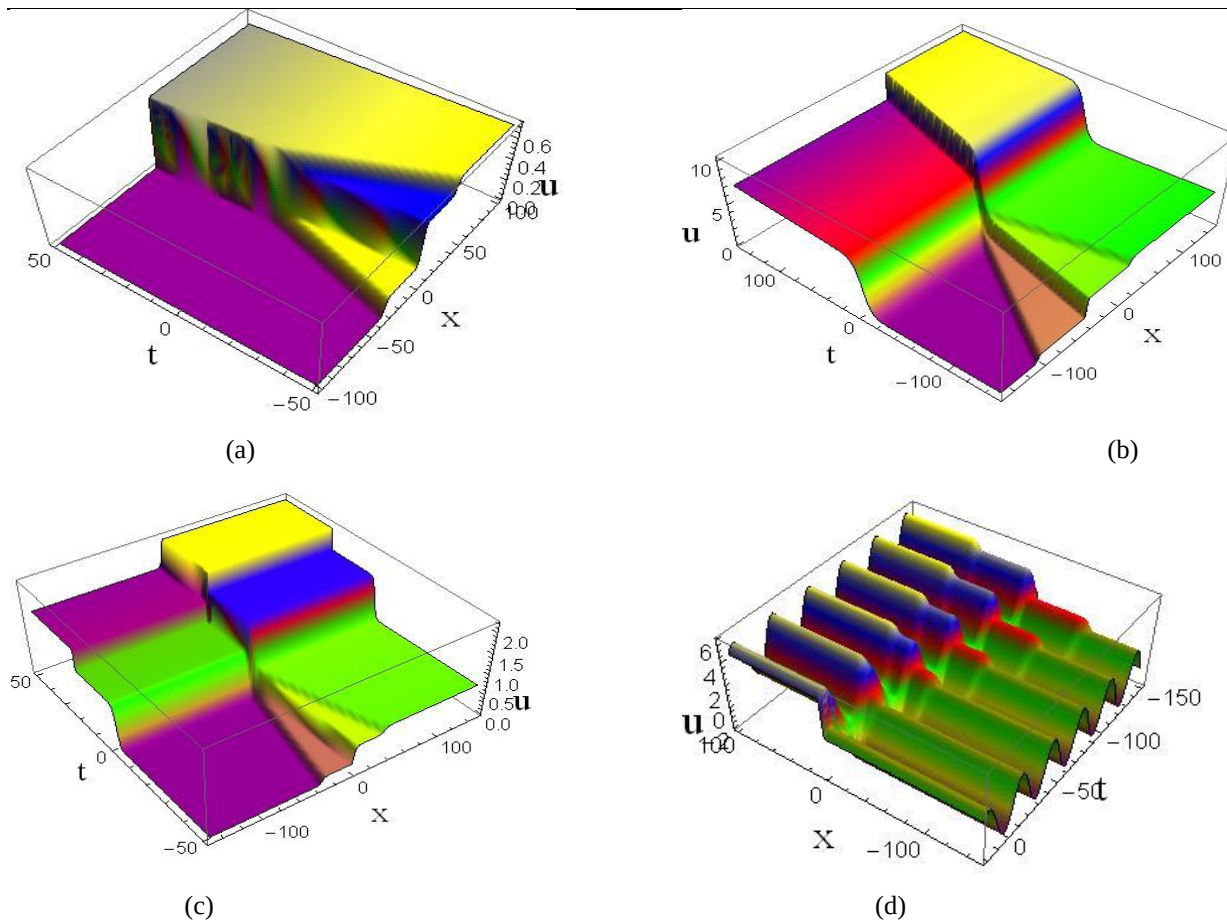


Fig. 4 3D profile of the 3-kink soliton solution of the Eq. (1), with the numeric choice of the parameters $c_1 = 0.1, A = 1.2, B = 0.25, C = 0.001, D = 0.8, k_1 = 0.8, k_2 = 2.8, k_3 = 3.5, \delta_1 = 1, \delta_2 = 2, \delta_3 = 3$, (a) $g = 0, w = 0.8, f(t) = 0$; (b) $g = 0.4, w = 0.4, f(t) = g \operatorname{sech}(wt)$; (c) $g = 0.4, w = 0.8, f(t) = g \operatorname{sech}^2(wt)$; (d) $g = 0.4, w = 0.2, f(t) = g \sin(wt)$.

II. Graphical presentation and Discussion

The Burger's equation is a fundamental model that captures the intricate dynamics of nonlinear systems, particularly the interplay between nonlinear convection (uu_x) and dissipative effects (u_{xx}). Originally introduced by Burger, this equation aims to elucidate the characteristics of turbulent fluid flow in a channel, where convection and diffusion exert opposing influences. The solution to this equation typically involves various parameters that play a crucial role in shaping wave formation. Analyzing the significance and role of these parameters is essential to understanding the underlying dynamics. By examining how these parameters influence the solution, one can gain valuable insights into the mechanisms driving wave behavior, including the formation of shock waves, solitons, and other nonlinear phenomena.

Fig.1 presented the two dimensional 2-kink type soliton solution given by equation (9) with suitable choice of the parameters. Fig.1(a) is the 2D graph of 2-kink soliton of equation (1) in absence of forcing terms i.e. $f(t)=0$. It is look like one stairs. Fig.1(b) presented 2D graph of 2-kink soliton of equation (1) in presence of force term $f(t) = g \operatorname{sech}(wt)$. It is observed that when force is increases, then the wave gets more amplitude. Fig.1(c) presented 2D graph of 2-kink soliton of equation (1) in presence of force term $f(t) = g \operatorname{sech}(wt)^2$ with different energy. Fig.1(d) presented 2D graph of 2-kink periodic soliton of equation (1) in presence of periodic force term $f(t)= g \sin(wt)$.

Fig.2 demonstrated the 3D version of the 2-kink type soliton solution given by the equation (9) with different choices of force terms (a) $f(t)=0$, (b) $f(t) = g \operatorname{sech}(wt)$, (c) $f(t) = g \operatorname{sech}(wt)^2$, (d) $f(t)= g \sin(wt)$.

Fig.3 depicted the two dimensional 3-kink type soliton solution given by equation (10) with suitable choice of the parameters. Fig.3(a) is the 2D graph of 3-kink soliton of equation (1) in absence of forcing terms i.e. $f(t)=0$. It is look like two stairs. Fig.3(b) presented 2D graph of 3-kink soliton of equation (1) in presence of force term $f(t) = g \operatorname{sech}(wt)$. It is observed that when force is increases, then the wave gets more amplitude. Fig.3(c) presented 2D graph of 3-kink soliton of equation (1) in presence of force

term $f(t) = g \operatorname{sech}(wt)^2$ with different energy. Fig.1(d) presented 2D graph of 3-kink periodic soliton of equation (1) in presence of periodic force term $f(t) = g \sin(wt)$.

Fig.4 demonstrated the 3D version of the 3-kink type soliton solution given by the equation (10) with different choices of force terms (a) $f(t)=0$, (b) $f(t) = g \operatorname{sech}(wt)$, (c) $f(t) = g \operatorname{sech}(wt)^2$, (d) $f(t) = g \sin(wt)$.

III. Conclusion

This paper presents the successful application of the simplified Hirota's method to obtain solutions for the nonlinear non-autonomous Burger's equation. The method yields new types of traveling wave solutions, including multi-shock solutions, which are graphically demonstrated with specific numerical choices for the constant parameters. The impact of the forcing term on the solution is visualized through 2D and 3D graphs, providing valuable insights into the equation's behavior.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

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