

Spectral Collocation Method for Generalized Abel's Integral Equations

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Abstract: This paper is devoted to the spectral collocation method using Jacobi polynomials for Generalized Abel's integral equations (GAIEs). Jacobi polynomials are used to approximate the solution which reduces GAIEs to a system of algebraic equations. To show the theoretical accuracy of the proposed method, convergence analysis has been proved. Also, the error estimate is approximated which shows that the approximation will converge to the exact solution. Two numerical examples are considered to verify and validate the theoretical results. Further, numerical results are compared using some particular cases of Jacobi polynomials.

Keywords: Jacobi polynomials; spectral method; Abel's integral equations.

I. Introduction

In this paper, we intend to consider a sort of GAIEs [1] which is given by,

$$\varphi(\rho) \int_0^\rho U(\tau)(\rho - \tau)^{-\theta} d\tau + \phi(\rho) \int_\rho^1 U(\tau)(\tau - \rho)^{-\theta} d\tau = V(\rho) \quad (1.1)$$

where, $0 < \theta < 1$, $\rho \in (0,1)$, and $\varphi(\rho), \phi(\rho), V(\rho)$ are given functions. GAIEs can be defined as first kind Fredholm integral equation (FIE) with convolution type weakly singular kernel. For $\phi(\rho) = 0$ and $\varphi(\rho) = 1$, the above equation takes the form

$$\int_0^\rho U(\tau)(\rho - \tau)^{-\theta} d\tau = V(\rho), 0 < \theta < 1, \rho \in (0,1), \quad (1.2)$$

This was introduced and studied by Abel [2] where he found the path of a particle in a vertical plane such that a mass particle takes the same time (no matter where the particle is placed) to slide under influence of gravity, reaches to the end of the curve at the same time. In 1896, Volterra [3] further studied such equations under the regularity condition. These equations are named as Volterra integral equations.

A few numerical strategies have researched for Eq. (1.1). For instance, Chakrabarty et al. [1] concluded a formula for the analytic solution of GAIEs. He converted the GAIE into a singular FIE of the Cauchy type utilizing some standard definitions of fractional calculus. He applied the same strategy to solve the system of GAIEs. In 2008 [4], Chakrabarty gave a direct function approach to figure out a solution close to the exact solution of GAIEs. Sandeep et al. coupled up to the Bernstein polynomials and almost operational matrix for the solution of GAIEs [5]. As of late [6], Pandey et al. used collocation method dependent to get the solution of GAIEs.

The fundamental target of this paper is to show a numerical technique for GAIEs. The introduced strategy depends on Jacobi spectral method combined with collocation nodes which are characterized as the roots of Jacobi polynomial. Numerous authors have been used spectral method effectively for solving diverse sort of integral and differential equations [7-10].

The presented method in this paper is based on the spectral method for solving the GAIEs Eq. (1.1). In Section 2, we describe the spectral collocation method. Section 3, contains result of the convergence analysis and error estimate. In Section 4, we show the numerical precision of the proposed method with numerical accuracy using different cases of Jacobi polynomials and the last area 5 finishes up the paper with conclusion.

II. Spectral Collocation Method

In the whole method and analysis we assumed that the operator defined on right hand side is one-one and onto. It is clear from [9] that operator on right hand side of Eq. (1.1) is bounded also in $\mathcal{L}^2$ with the bound

$$\begin{aligned} & \left\| \int_0^\rho U(\tau)(\rho - \tau)^{-\theta} d\tau + \int_\rho^1 U(\tau)(\tau - \rho)^{-\theta} d\tau \right\|_{\mathcal{L}^2} \leq \left\| \int_0^\rho U(\tau)(\rho - \tau)^{-\theta} d\tau \right\|_{\mathcal{L}^2} \\ & + \left\| \int_\rho^1 U(\tau)(\tau - \rho)^{-\theta} d\tau \right\|_{\mathcal{L}^2} \leq \frac{2\Gamma(1-\theta)}{\Gamma(2-\theta)} \|U\|_{\mathcal{L}^2}. \end{aligned}$$

Let $\Omega = [-1,1]$ and let P_N represents the polynomial space of degree N and $j_{\alpha,\beta}^N(\rho)$ denote the Jacobi polynomial [9] of degree N , given by,

$$\frac{\Gamma(a+N+1)}{N!\Gamma(a+b+N+1)} \sum_{i=0}^N \binom{N}{i} \frac{\Gamma(a+b+N+i+1)}{\Gamma(a+i+1)} \left(\rho - \frac{1}{2}\right)^i.$$

Jacobi polynomial forms an orthogonal basis for P_N with respect to the weight function,

$$\omega^{\alpha,\beta}(\rho) = (1-\rho)^\alpha(1+\rho)^\beta, \quad -1 < \alpha, \beta < 1. \quad (2.1)$$

Let $\gamma_k^{\alpha,\beta}$ be the node points of Gauss-Jacobi quadrature (GJQ) formula, given by,

$$j_{\alpha,\beta}^{N+1}(\gamma_k^{\alpha,\beta}) = 0, \quad k = 0, 1 \dots N, \quad (2.2)$$

and let $\omega_k^{\alpha,\beta}$ denote the weight function of GJQ formula, $k = 0, 1 \dots N$.

For using the orthogonal property of Jacobi polynomials, First we apply a variable transformation $x \rightarrow \frac{x+1}{2}$, to shift the above equation from $[0,1]$ to $[-1,1]$ (without lose of generality we assume $\varphi(\rho) = \phi(\rho) = 1$),

$$\rho = \frac{1+z}{2}, \quad \tau = \frac{1+w}{2}, \quad (2.3)$$

and we obtain,

$$\frac{1}{2^{1-\theta}} \left(\int_{-1}^z U\left(\frac{1+w}{2}\right) (z-w)^{-\theta} dw + \int_z^1 U\left(\frac{1+w}{2}\right) (w-z)^{-\theta} dw \right) = V\left(\frac{1+z}{2}\right), \quad (2.4)$$

by letting $u(w) = U\left(\frac{1+w}{2}\right)$, $v(z) = 2^{1-\theta} V\left(\frac{1+z}{2}\right)$, we have

$$\int_{-1}^z u(w)(z-w)^{-\theta} dw + \int_z^1 u(w)(w-z)^{-\theta} dw = v(z). \quad (2.5)$$

Now, we seek a function written as $u_N(\rho) = \sum_{i=0}^N \xi_i j_{\alpha,\beta}^i(\rho) \in \mathcal{P}_N$, such that for $0 \leq k \leq N$,

$$\int_{-1}^{\gamma_k^{\alpha,\beta}} u_N(w)(\gamma_k^{\alpha,\beta} - w)^{-\theta} dw + \int_{\gamma_k^{\alpha,\beta}}^1 u_N(w)(w - \gamma_k^{\alpha,\beta})^{-\theta} dw = v(\gamma_k^{\alpha,\beta}), \quad (2.6)$$

$$J_1 + J_2 = v(\gamma_k^{\alpha,\beta}). \quad (2.7)$$

Where, $J_1 = \int_{-1}^{\gamma_k^{\alpha,\beta}} u_N(w)(\gamma_k^{\alpha,\beta} - w)^{-\theta} dw$, and $J_2 = \int_{\gamma_k^{\alpha,\beta}}^1 u_N(w)(w - \gamma_k^{\alpha,\beta})^{-\theta} dw$.

By applying a linear variable transformation,

$$w = \varepsilon_k(\delta) = \frac{1+\gamma_k^{\alpha,\beta}}{2} \delta + \frac{\gamma_k^{\alpha,\beta}-1}{2}, \quad (2.8)$$

J_1 converts into,

$$\begin{aligned} J_1 &= \int_{-1}^{\gamma_k^{\alpha,\beta}} u_N(w)(\gamma_k^{\alpha,\beta} - w)^{-\theta} dw, \\ &= \int_{-1}^1 u_N(\varepsilon_k(\delta))(1-\delta)^{-\theta} \left(\frac{\gamma_k^{\alpha,\beta}+1}{2} \right)^{1-\theta} d\delta. \end{aligned} \quad (2.9)$$

$$\text{Similarly, by } w = \varepsilon_k(\delta) = \frac{1-\gamma_k^{\alpha,\beta}}{2} \delta + \frac{\gamma_k^{\alpha,\beta}+1}{2}, \quad (2.10)$$

J_2 reduces to,

$$J_2 = \int_{-1}^1 u_N(\varepsilon_k(\delta))(\delta+1)^{-\theta} \left(\frac{1-\gamma_k^{\alpha,\beta}}{2} \right)^{1-\theta} d\delta. \quad (2.11)$$

Using Eq. (2.9) and (2.11) in Eq. (2.6) we get,

$$\int_{-1}^1 u_N(\varepsilon_k(\delta))(1-\delta)^{-\theta} \left(\frac{\gamma_k^{\alpha,\beta}+1}{2} \right)^{1-\theta} d\delta + \int_{-1}^1 u_N(\varepsilon_k(\delta))(\delta+1)^{-\theta} \left(\frac{1-\gamma_k^{\alpha,\beta}}{2} \right)^{1-\theta} d\delta = v(\gamma_k^{\alpha,\beta}),$$

After applying the GJQ formula with corresponding weight function for J_1, J_2 , the above equation reduces to,

$$\sum_{r=0}^N \left(\frac{\gamma_k^{\alpha,\beta}+1}{2} \right)^{1-\theta} u_N(\varepsilon_k(\delta_{1_r})) \zeta_{1_r} + \sum_{r=0}^N \left(\frac{1-\gamma_k^{\alpha,\beta}}{2} \right)^{1-\theta} u_N(\varepsilon_k(\delta_{2_r})) \zeta_{2_r} = v(\gamma_k^{\alpha,\beta}). \quad (2.12)$$

here, $\delta_{1_r} = \gamma_r^{-\theta,0}$, $\zeta_{1_r} = \omega_r^{-\theta,0}$ and $\delta_{2_r} = \gamma_r^{0,-\theta}$, $\zeta_{2_r} = \omega_r^{0,-\theta}$, $k = 0, 1, \dots N$.

Some Useful Lemmas

This section includes some basic definition and results which will be used throughout the paper which will be helpful for the error analysis.

. Let $\mathcal{L}_{\omega^{\alpha,\beta}}^2(\Omega)$ be the space of all square Lebesgue integrable and measurable function defined over Ω with weight $\omega^{\alpha,\beta}$. We define the continuous inner product on $\mathcal{L}_{\omega^{\alpha,\beta}}^2(\Omega)$ as,

$$\langle u|v \rangle_{\omega^{\alpha,\beta}} = \int_{-1}^1 u(\rho) v(\rho) \omega^{\alpha,\beta}(\rho) d\rho, \quad (3.1)$$

and the norm is given by

$$\|u\|_{\omega^{\alpha,\beta}} = \int_{-1}^1 (u(\rho))^2 \omega^{\alpha,\beta}(\rho) d\rho. \quad (3.2)$$

Now, we introduce orthogonal projection $\Pi_N: \mathcal{L}_{\omega^{\alpha,\beta}}^2 \rightarrow \mathcal{P}_N$ which satisfies,

$$\langle \Pi_N u | v_N \rangle_{\omega^{\alpha,\beta}} = \langle u | v_N \rangle_{\omega^{\alpha,\beta}}, \quad \forall u \in \mathcal{L}_{\omega^{\alpha,\beta}}^2, v_N \in \mathcal{P}_N. \quad (3.3)$$

Also, we define

$$\mathcal{H}_{\omega^{\alpha,\beta}}^l(\Omega) = \{u: \mathcal{D}^r u \in \mathcal{L}_{\omega^{\alpha,\beta}}^2(\Omega), 0 \leq r \leq l\}, \quad (3.4)$$

with the norm

$$\|u\|_{\mathcal{H}_{\omega^{\alpha,\beta}}^l(\Omega)} = \left(\sum_{r=0}^l \|\mathcal{D}^r u\|_{\omega^{\alpha,\beta}}^2 \right)^{\frac{1}{2}}. \quad (3.5)$$

Lemma 1 For any $u \in \mathcal{H}_{\omega^{\alpha,\beta}}^l(\Omega)$ and $l \leq 1$, we have

$$\|u - \Pi_N u\|_{\omega^{\alpha,\beta}} \leq CN^{-l} |u|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)}, \quad (3.6)$$

$$\|u - \Pi_N u\|_{\infty} \leq CN^{\frac{3}{4}-l} |u|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)}, \quad (3.7)$$

where $|u|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)}$ is the seminorm which is defined by

$$|u|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)} = \left(\sum_{r=\min(l,N+1)}^l \|\mathcal{D}^r u\|_{\omega^{\alpha,\beta}}^2 \right)^{\frac{1}{2}}. \quad (3.8)$$

Lemma 2 For any $u \in \mathcal{L}_{\omega^{\alpha,\beta}}^2(\Omega)$, we have

$$\|\Pi_N u\|_{\omega^{\alpha,\beta}} \leq C \|u\|_{\omega^{\alpha,\beta}}, \quad (3.9)$$

$$\|\Pi_N u\|_{\infty} \leq C \|u\|_{\infty}. \quad (3.10)$$

Lemma 3 Suppose we approximate the integration of uv by the $N + 1$ Gauss quadrature formula relative to the Jacobi weight function, where $u \in \mathcal{H}_{\omega^{\alpha,\beta}}^l(\Omega)$ for any $1 \leq l$ and $v \in \mathcal{P}_N$. Then we have a constant C which does not depend on N such that

$$|\langle u|v \rangle_{\omega^{\alpha,\beta}} - \langle u|v \rangle_N| \leq CN^{-l} |u|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)} \|v\|_{\omega^{\alpha,\beta}}. \quad (3.11)$$

Convergence Analysis

Theorem 1: Let $\mathcal{M}$ be the integral operator on $\mathcal{L}_{\omega^{\alpha,\beta}}^2(\Omega)$ defined by

$$\mathcal{M}(u) = \int_{-1}^1 \mathcal{G}_1(\gamma_k^{\alpha,\beta}, \delta) u(\varepsilon_k(\delta)) d\delta + \int_{-1}^1 \mathcal{G}_2(\gamma_k^{\alpha,\beta}, \delta) u(\epsilon_k(\delta)) d\delta, \quad (3.12)$$

where $\varepsilon_k(\delta), \epsilon_k(\delta)$ are defined by Eq. (2.8) and Eq. (2.10) respectively. Let $\mathcal{M}_N$ be the GJQ approximation of $\mathcal{M}$ with respect to the weight function $\gamma_k^{\alpha,\beta}$. Then we have,

$$|\mathcal{M}(u_N) - \mathcal{M}_N(u_N)| \leq CN^{-l} C^* \|u_N\|_{\omega^{\alpha,\beta}}. \quad (3.13)$$

Proof: we can write $\mathcal{M}(u_N)$ and $\mathcal{M}_N(u_N)$ in the form of inner product as,

$$\mathcal{M}(u_N) = \left\langle \mathcal{G}_1(\gamma_k^{\alpha,\beta}, \delta) | u_N(\varepsilon_k(\delta)) \right\rangle_{\gamma_k^{\alpha,\beta}} + \left\langle \mathcal{G}_2(\gamma_k^{\alpha,\beta}, \delta) | u_N(\epsilon_k(\delta)) \right\rangle_{\gamma_k^{\alpha,\beta}}, \quad (3.14)$$

$$\text{and } \mathcal{M}_N(u_N) = \left\langle \mathcal{G}_1(\gamma_k^{\alpha,\beta}, \delta) | u_N(\varepsilon_k(\delta)) \right\rangle_N + \left\langle \mathcal{G}_2(\gamma_k^{\alpha,\beta}, \delta) | u_N(\epsilon_k(\delta)) \right\rangle_N. \quad (3.15)$$

$$\begin{aligned} \mathcal{M}(u_N) - \mathcal{M}_N(u_N) &= \left\langle \mathcal{G}_1(\gamma_k^{\alpha,\beta}, \delta) | u_N(\varepsilon_k(\delta)) \right\rangle_{\gamma_k^{\alpha,\beta}} - \left\langle \mathcal{G}_1(\gamma_k^{\alpha,\beta}, \delta) | u_N(\varepsilon_k(\delta)) \right\rangle_N \\ &+ \left\langle \mathcal{G}_2(\gamma_k^{\alpha,\beta}, \delta) | u_N(\varepsilon_k(\delta)) \right\rangle_{\gamma_k^{\alpha,\beta}} - \left\langle \mathcal{G}_2(\gamma_k^{\alpha,\beta}, \delta) | u_N(\varepsilon_k(\delta)) \right\rangle_N, \end{aligned} \quad (3.16)$$

Using lemma (3), we get,

$$\begin{aligned} |\mathcal{M}(u_N) - \mathcal{M}_N(u_N)| &\leq CN^{-l} \left(|\mathcal{G}_1(\gamma_k^{\alpha,\beta}, \delta)|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)} + |\mathcal{G}_2(\gamma_k^{\alpha,\beta}, \delta)|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)} \right) \\ &\leq CN^{-l} C^* \|u_N\|_{\omega^{\alpha,\beta}}. \end{aligned} \quad (3.17)$$

$$\text{where } C^* = |\mathcal{G}_1(\gamma_k^{\alpha,\beta}, \delta)|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)} + |\mathcal{G}_2(\gamma_k^{\alpha,\beta}, \delta)|_{\mathcal{H}_{\omega^{\alpha,\beta}}^{l,N}(\Omega)}. \quad (3.18)$$

Error estimates:

We can Eq. (1.1) using Eq. (3.12) as,

$$\mathcal{M}(u(\rho)) = v(\rho). \quad (3.19)$$

We seek a function $u_N \in P_N$ such that,

$$\mathcal{M}_N u_N(\rho) = v(\rho). \quad (3.20)$$

Now from Eq. (3.19) we have,

$$u(\rho) = \mathcal{M}^{-1} v(\rho), \quad (3.21)$$

From Eq. (3.20), we obtain,

$$\begin{aligned} u(\rho) &= \mathcal{M}^{-1} \mathcal{M}_N u_N(\rho) = \mathcal{M}^{-1} (\mathcal{M} + \mathcal{M}_N - \mathcal{M}) u_N(\rho), \\ u(\rho) &= u_N(\rho) + \mathcal{M}^{-1} (\mathcal{M}_N - \mathcal{M}) u_N(\rho), \\ u(\rho) - u_N(\rho) &= \mathcal{M}^{-1} (\mathcal{M}_N - \mathcal{M}) u_N(\rho), \\ \|u - u_N\| &\leq \|\mathcal{M}^{-1}\| \|(\mathcal{M}_N - \mathcal{M}) u_N\|. \end{aligned} \quad (3.22)$$

Using Eq. (31) we obtain

$$\|u - u_N\| \leq CN^{-l} C^* \|\mathcal{M}^{-1}\| \|u_N\|_{\omega^{\alpha,\beta}}. \quad (3.23)$$

$$\text{It is clear from Eq. (3.23), } u \rightarrow u_N \text{ as } N \rightarrow \infty. \quad (3.24)$$

Numerical Experiments

In this section, the efficiency of the proposed method will be demonstrated with the numerical examples.

Test Example 1 We consider GAIEs (1) from [5, 6] with $\theta = 1/5$ and

$$v(\rho) = e^\rho \left[(1-\rho)^{4/5} (\rho-1)^{-4/5} \left\{ \Gamma\left(\frac{4}{5}\right) - \Gamma\left(\frac{4}{5}, \rho-1\right) \right\} + \left\{ \Gamma\left(\frac{4}{5}\right) - \Gamma\left(\frac{4}{5}, \rho\right) \right\} \right],$$

where Γ denotes the Gamma function. This problem has the solution $u(\rho) = e^\rho$.

We study the performance of spectral collocation scheme for this problem. Table 1, Table 2, Table 3 and Table 4 show the L^2 and L^∞ errors obtained with $\alpha = \beta = 0$ (Legendre polynomials), $\alpha = \beta = -0.5$ (Chebyshev polynomials), $\alpha = 0.4, \beta = -0.4$ and $\alpha = 0.8, \beta = -0.4$ respectively. Comparison of numerical and exact solution is given in Table 5 and Table 6 with $\alpha = \beta = 0$ (Legendre polynomials), $\alpha = \beta = -\frac{1}{2}$ (Chebyshev polynomials). The variation of the L^2 and L^∞ errors for different values α, β versus N are shown in Fig.1 and Fig.2 respectively.

Test Example 2 Consider the GAIE [5, 6] given by Eq.(1) with $\theta = 1/2$ and $v(\rho) = \frac{4}{105} \rho^{3/2} (35 - 24\rho^2) + \frac{8}{105} (1 - \rho)^{1/2} (5 + 13\rho - 6\rho^2 + 12\rho^3)$. This problem has exact solution $u(\rho) = \rho - \rho^3$.

We solve this problem by the method discussed in section 2 with different values of N . In this case, we obtained the numerical solution with $N = 2, 3, 4, 5$ closer to the exact solution. The best numerical approximation of the solution are $0.25j_{0,0}^0(\rho) + 0.04999j_{0,0}^1(\rho) - 0.25j_{0,0}^2(\rho) - 0.05j_{0,0}^3(\rho)$, $0.1875j_{-0.5,-0.5}^0(\rho) + 0.03125j_{-0.5,-0.5}^1(\rho) - 0.18749j_{-0.5,-0.5}^2(\rho) - 0.03124j_{-0.5,-0.5}^3(\rho)$, $0.196j_{0.4,-0.4}^0(\rho) + 0.094j_{0.4,-0.4}^1(\rho) - 0.2167j_{0.4,-0.4}^2(\rho) - 0.05j_{0.4,-0.4}^3(\rho)$ and $0.1804j_{0.8,-0.4}^0(\rho) + 0.0989j_{0.8,-0.4}^1(\rho) - 0.1629j_{0.8,-0.4}^2(\rho) - 0.0394j_{0.8,-0.4}^3(\rho)$ by Legendre and Chebyshev polynomials respectively (for $N = 3$).

The illustration of the obtained L^2 and L^∞ errors are shown in Tables 7-10 and Figs. 5,6 with $\alpha = \beta = 0$ (Legendre polynomials), $\alpha = \beta = -0.5$ (Chebyshev polynomials), $\alpha = 0.4, \beta = -0.4$ and $\alpha = 0.8, \beta = -0.4$.

Table-1. The errors of spectral collocation method with $\alpha = \beta = 0$ for example 1.

N	2	3	4	5	6	7
L^∞	$1.597E-2$	$1.288E-3$	$8.292E-5$	$2.825E-6$	$1.092E-7$	$3.604E-9$
L^2	$2.849E-2$	$2.138E-3$	$1.287E-4$	$4.585E-6$	$1.749E-7$	$5.804E-9$

Table-2. The errors of spectral collocation method with $\alpha = \beta = -0.5$ for example 1.

N	2	3	4	5	6	7
L^∞	$1.344E-2$	$8.730E-4$	$4.411E-5$	$1.864E-6$	$6.685E-8$	$2.104E-9$
L^2	$2.761E-2$	$1.799E-3$	$9.313E-5$	$4.005E-6$	$1.462E-7$	$4.631E-9$

Table-3. The errors of spectral collocation method with $\alpha = 0.4, \beta = -0.4$ for example 1.

N	2	3	4	5	6	7
L^∞	$2.399E-2$	$1.893E-3$	$1.131E-4$	$5.502E-6$	$2.235E-7$	$7.826E-9$
L^2	$3.390E-2$	$2.394E-3$	$1.344E-4$	$6.292E-6$	$2.510E-7$	$8.692E-9$

Table-4. The errors of spectral collocation method with $\alpha = 0.8, \beta = -0.4$ for example 1.

N	2	3	4	5	6	7
L^∞	$2.853E-2$	$2.016E-3$	$1.569E-4$	$9.493E-6$	$5.211E-7$	$2.377E-8$
L^2	$3.892E-2$	$3.014E-3$	$2.233E-4$	$1.320E-5$	$7.148E-7$	$3.249E-8$

Table-5. Numerical solution versus exact $\alpha = \beta = 0$ (Legendre polynomials) for example 1.

N	2	3	4	5	6	7	Exact
0.0	1.0129	0.9988	0.9999	0.9999	1.0	1.0	1.0
0.1	1.1066	1.1052	1.1052	1.1051	1.1051	1.1051	1.1051
0.2	1.2170	1.2216	1.2214	1.2214	1.2214	1.2214	1.2214
0.3	1.3441	1.3498	1.3498	1.3498	1.3498	1.3498	1.3498
0.4	1.4880	1.4914	1.4918	1.4918	1.4918	1.4918	1.4918
0.5	1.6486	1.6482	1.6487	1.6487	1.6487	1.6487	1.6487
0.6	1.8285	1.8217	1.8220	1.8221	1.8221	1.8221	1.8221
0.7	2.0199	2.0137	2.0137	2.0137	2.0137	2.0137	2.0137
0.8	2.2306	2.2258	2.2255	2.2255	2.2255	2.2255	2.2255
0.9	2.4581	2.4596	2.4596	2.4596	2.4596	2.4596	2.4596
1.0	2.7023	2.7169	2.7182	2.7182	2.7182	2.7182	2.7182

Table-6. Numerical solution versus exact $\alpha = \beta = -0.5$ (Chebyshev polynomials) for example 1.

N	2	3	4	5	6	7	Exact
0.0	1.0109	0.9992	0.9994	0.9998	1.0	1.0	1.0
0.1	1.1049	1.1054	1.1052	1.051	1.1051	1.1051	1.1051
0.2	1.2157	1.2238	1.2224	1.2218	1.2214	1.2214	1.2214
0.3	1.3432	1.3499	1.3498	1.3498	1.3498	1.3498	1.3498

0.4	1.4874	1.4915	1.4918	1.4918	1.4918	1.4918	1.4918
0.5	1.6485	1.6482	1.6487	1.6487	1.6487	1.6487	1.6487
0.6	1.8262	1.8218	1.8221	1.8221	1.8221	1.8221	1.8221
0.7	2.0207	2.0138	2.0137	2.0137	2.0137	2.0137	2.0137
0.8	2.2320	2.2259	2.2255	2.2255	2.2255	2.2255	2.2255
0.9	2.4600	2.4599	2.4596	2.4596	2.4596	2.4596	2.4596
1.0	2.7048	2.7174	2.7182	2.7182	2.7182	2.7182	2.7182

Table-7. The errors of spectral collocation method with $\alpha = \beta = 0$ for example 2.

N	2	3	4	5
L^∞	$5.088E - 2$	$1.221E - 16$	$7.494E - 16$	$1.665E - 16$
L^2	$1.012E - 1$	$1.963E - 16$	$1.033E - 15$	$4.119E - 16$

Table-8. The errors of spectral collocation method with $\alpha = \beta = -0.5$ for example 2.

N	2	3	4	5
L^∞	$5.625E - 1$	$2.498E - 16$	$3.885E - 16$	$7.563E - 16$
L^2	$1.281E - 1$	$5.157E - 16$	$7.422E - 16$	$1.452E - 15$

Table-9. The errors of spectral collocation method with $\alpha = 0.4, \beta = -0.4$ for example 2.

N	2	3	4	5
L^∞	$8.520E - 2$	$1.665E - 16$	$3.885E - 16$	$1.924E - 15$
L^2	$1.230E - 1$	$7.814E - 16$	$9.814E - 16$	$2.314E - 15$

Table-10. The errors of spectral collocation method with $\alpha = 0.8, \beta = -0.4$ for example 2.

N	2	3	4	5
L^∞	$7.927E - 2$	$6.938E - 16$	$2.220E - 15$	$6.772E - 15$
L^2	$1.414E - 1$	$2.196E - 15$	$3.269E - 15$	$9.033E - 15$

Comparison with the existing methods for Generalized Abel's integral equation-

The above test examples are taken from the [5, 6]. In [5], the solution of generalized Abel's integral equation is obtained via operational matrix by taking Bernstein polynomials as basis function. In the formation of operational matrix, the continuous integral term is calculated. Also, only numerical experiment is presented in that paper, no result related to theoretical accuracy or convergence of the presented method had been presented. In [6], a direct approach based on Jacobi polynomials and collocation method is presented and an error bound is provided for the presented approach. In this approach [6], the unknown term is approximated in terms of Jacobi polynomials and by substituting and calculating the integral term, numerical solution is obtained. In both methods, the integral term is calculated. Our method is based on discretization of the integral by Gauss-Jacobi quadrature rule which converts the integral into summation and the integral term need not to calculate. In fact, it reduces the absolute error on the boundary in comparison of [5]. The obtained maximum absolute error for test example 1 and test example 2 in [5] have $O(10^{-4})$ and $O(10^{-6})$ respectively. In [6], it is of order $O(10^{-7})$ for test example 1 and $O(10^{-16})$ for test example 2. In addition, convergence in the weighted L^2 norm and L^∞ norm is provided.

Table 11. Comparison of spectral collocation method with existing methods for test example 1.

t	Exact value of $u(\rho) = e^\rho$	Method presented in section 2 for $N = 7$. $\alpha = \beta = 0$	Method presented in section 2 for $N = 7$. $\alpha = \beta = -0.5$	Method proposed in [5]
0.0	1	0.99999	0.99999	1.0002

0.2	1.22140	1.22140	1.22140	1.22145
0.4	1.49182	1.49182	1.49182	1.49183
0.6	1.82211	1.82211	1.82211	1.82210
0.8	2.22554	2.22554	2.22554	2.22558
1.0	2.71828	2.71828	2.71828	2.71828

Table 12. Comparison of spectral collocation method with existing methods for test example 2.

t	Exact value of $u(\rho) = \rho - \rho^3$	Method presented in section 2 for $N = 3$. $\alpha = \beta = 0$	Method presented in section 2 for $N = 3$. $\alpha = \beta = -0.5$	Method proposed in [5]
0.0	0	$9.0E - 16$	$-1.0E - 16$	$-1E - 6$
0.2	0.192	0.19199	0.19200	0.19199
0.4	0.336	0.33599	0.33600	0.33599
0.6	0.384	0.38400	0.38399	0.38400
0.8	0.288	0.28800	0.28800	0.28798
1.0	0	$-1E - 16$	$-1E - 16$	$-5E - 6$

III. Conclusions:

We propose a method based on Jacobi polynomials and spectral collocation method for GAIEs. This scheme is based on discretization which includes the Jacobi polynomials and roots of Jacobi polynomials to discretize the domain. We provide the results concerning theoretical accuracy as well numerical accuracy. L^∞ and L^2 error estimates are theoretically demonstrated for the proposed method and also validated by numerical example. Also we compare the results obtained with different cases of Jacobi polynomials. Further, it is presumed that proposal in this work will provide motivation for our future work on the non-linear singular integral equations. It is obvious that the presented method is more computationally effective and a direct approach than the method presented in [5] for GAIEs.

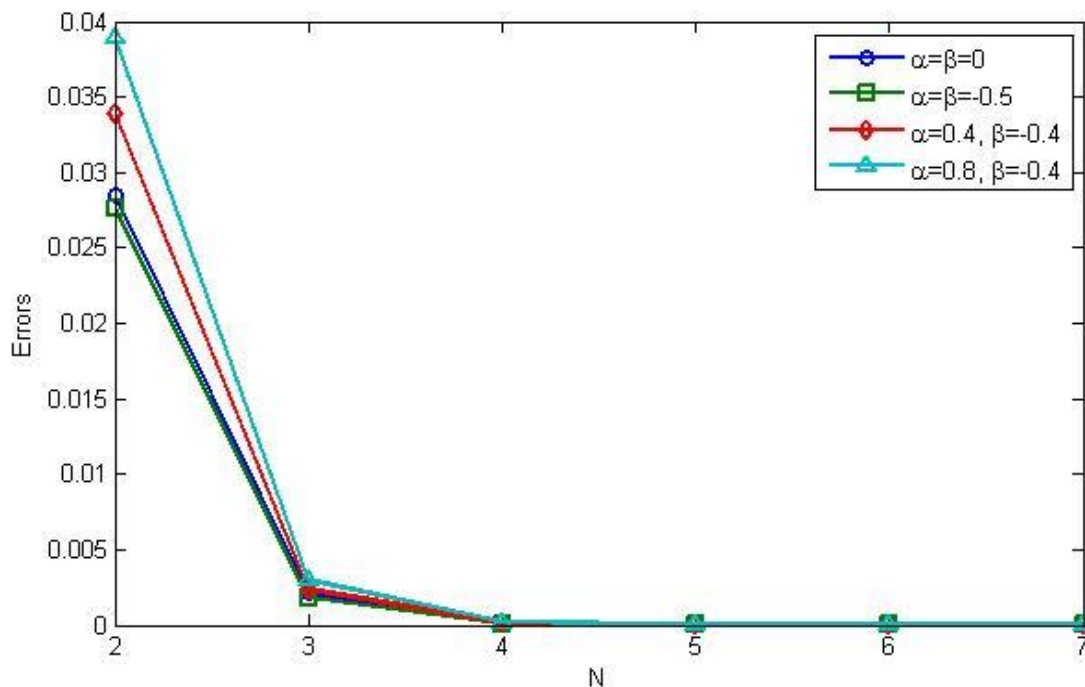


Fig. 1.a L^2 errors for example 1 versus N .

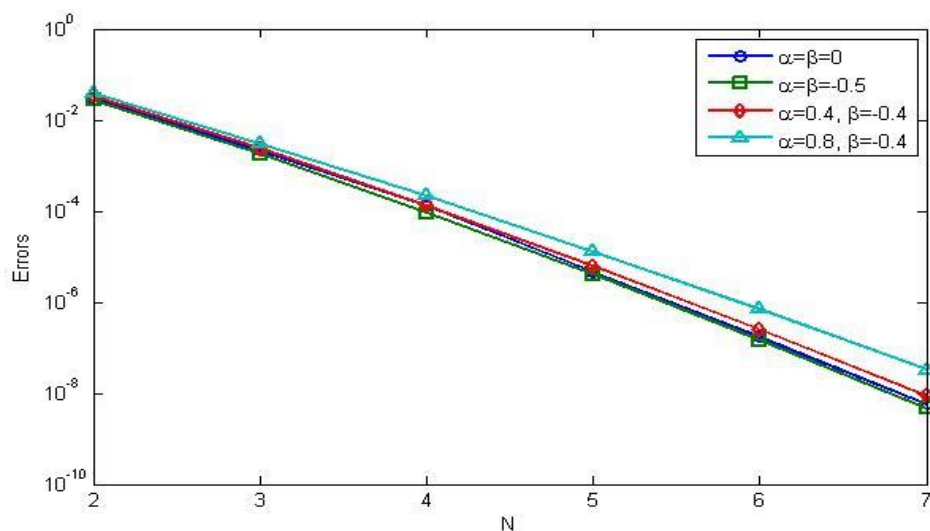


Fig. 1.b L^2 errors in log scale for example 1 versus N .

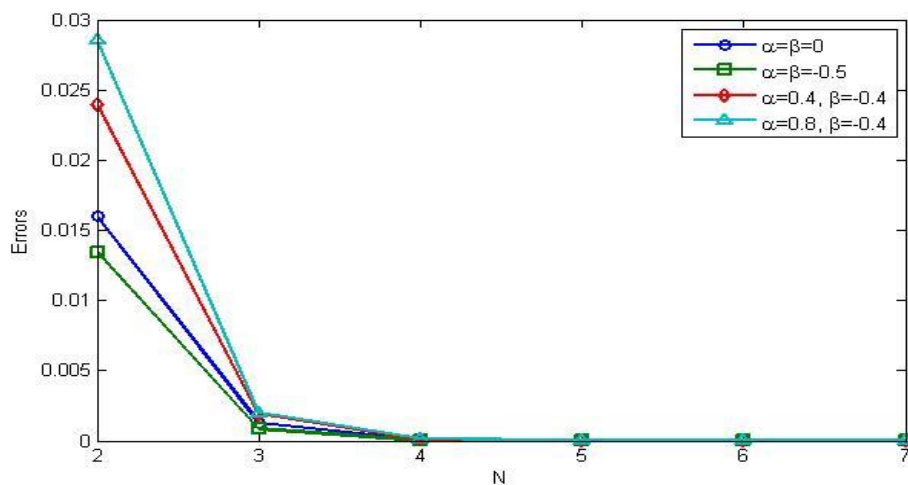


Fig. 2.a L^∞ errors for example 1 versus N .

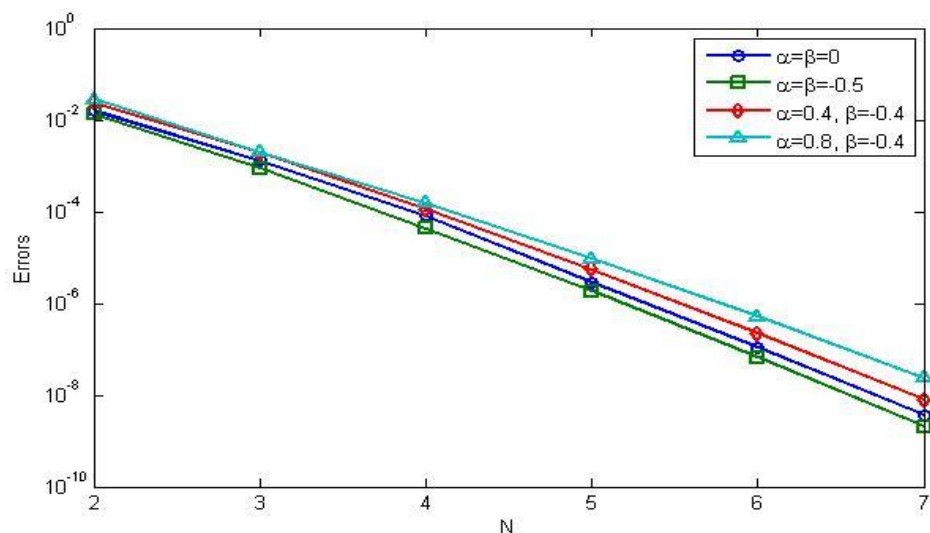


Fig. 2.b L^∞ errors in log scale for example 1 versus N .

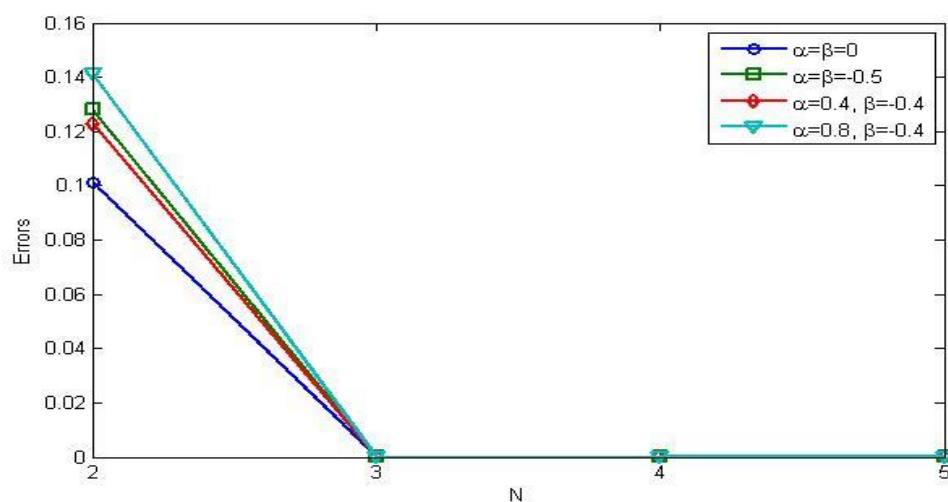


Fig. 3.a L^2 errors for example 2 versus N .

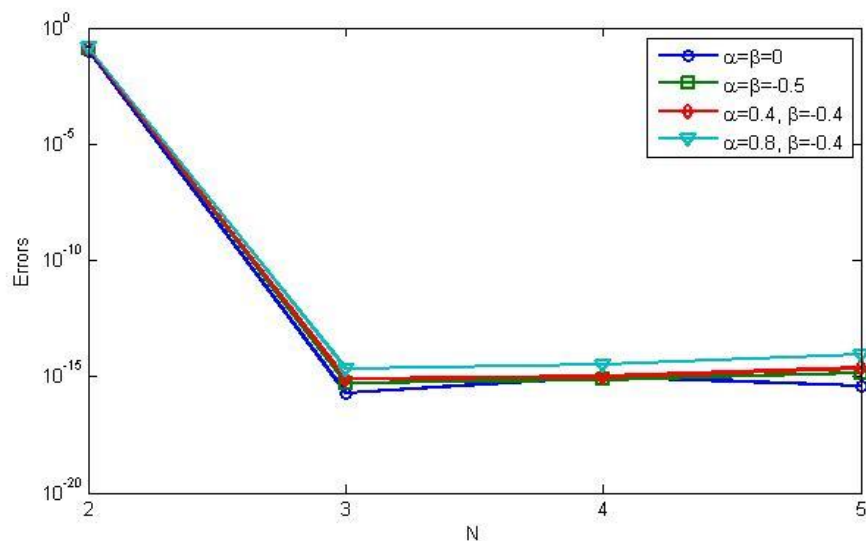


Fig. 3.b L^2 errors in log scale for example 2 versus N .

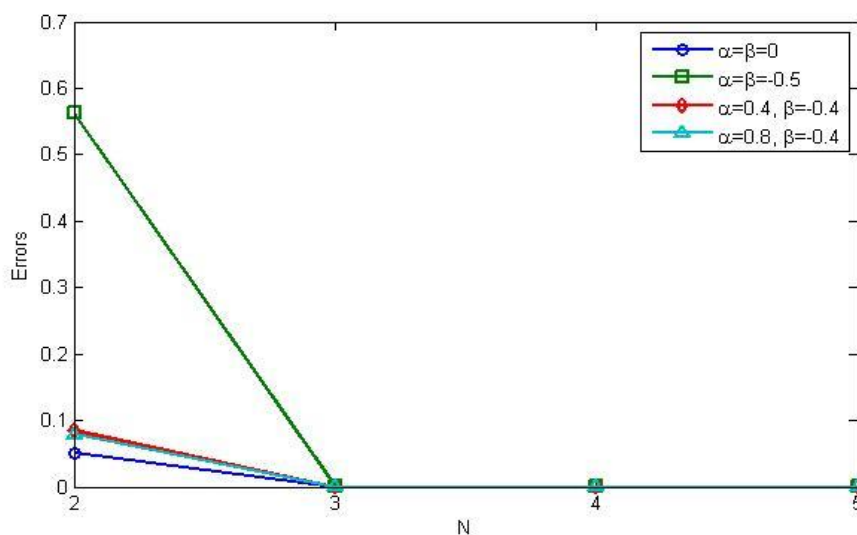


Fig. 4.a L^∞ errors for example 2 versus N .

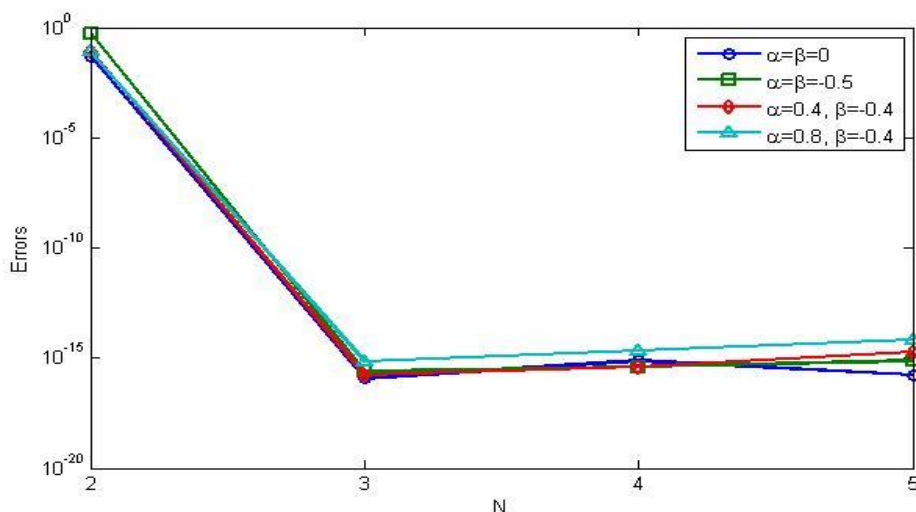


Fig. 4.b L^∞ errors in log scale for example 2 versus N .

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