

The Diophantine Exponent of Algebraic Numbers for β -Expansions with a Negative Base

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Abstract: The aim of this paper is to prove that the Diophantine exponent $Dio\left(d_{-\beta}(l_{\beta})\right)$ is bounded by $\frac{\log M(\beta)}{\log \beta} + (d-1)\frac{\log |a_d|+1}{\log \beta}$, where β is an algebraic number, and $M(\beta)$ is the Mahler measure of β . In addition, a new result will be founded about transcendental number a when $Dio(a)$ is infinite.

Key words: β -Expansion, Diophantine Approx- imation, Transcendental, Mahler Measure, Negative Base.

I. Introduction

The concept of β -expansion was introduced in (1957) by Renyi [1], and since then many researchers were interested to find a relationship between symbolic dynamics and the property of β -expansions, where β is an algebraic or real number greeter than 1. (e.g., [2, 9, 12]). β -expansions is related with the transformation $T : [0, 1) \rightarrow [0, 1)$ such that $T(x) := \beta x - \lfloor \beta x \rfloor$ and any $x \in [0, 1)$ is a sum of the following infinite series:

$$x_i = \sum_{i=1}^{\infty} \frac{x_i}{\beta^i}, \text{ where } x_i = \lfloor \beta T^{i-1}(x) \rfloor \text{ for all } i = 1, 2, 3, \dots \quad (*)$$

Now from the definition of T , we can conclude that ‘digits’ x_i take values in the set $A = \{0, 1, 2, \dots, \lfloor \beta \rfloor - 1\}$ for $i = 1, 2, 3, \dots$. The expression of x in the form $(*)$ is called the β -expansion of x . The number x is thus represented by the infinite word $d_{\beta}(x) = x_1 x_2 x_3 \dots \in A^{\mathbb{N}}$ (resp. $A^{\mathbb{Z}}$) over the alphabet $A = \{0, 1, 2, \dots, \lfloor \beta \rfloor - 1\}$, knowing that: ‘word’ means sequence of elements of an alphabet A .

Directly from $(*)$ we can derive that the ‘digits’ x_i takes values in the set A . The number x is thus represented by the infinite word $d_{\beta}(x) = x_1 x_2 x_3 \dots \in A^{\mathbb{N}}$ over the alphabet A . From the definition of the transformation T , we can derive for another important property, namely that the ordering on real numbers is carried over to the ordering of β -expansions. In particular, we have for any $x, y \in [0, 1)$ that:

$$x \leq y \Leftrightarrow d_{\beta}(x) \leq d_{\beta}(y),$$

when $\leq$ is the lexicographical order on $A^{\mathbb{N}}$, (ordering on the alphabet A is usual

$$0 < 1 < 2 < \dots < (\lfloor \beta \rfloor - 1).$$

In [13], Parry has provided criteria for deciding whether the infinite word in $A^{\mathbb{N}}$ is a β -expansion of some real number x . This criterion was formulated by using the so-called infinite expansion of 1 and denoted by $d_{\beta}^*(1)$, defined as a limit in the space $A^{\mathbb{N}}$ equipped with the product topology, by

$$d_{\beta}^*(1) = \lim_{\varepsilon \rightarrow 0+} d_{\beta}(1 - \varepsilon)$$

Recently, Adamczewski and Bugeaud in [4] investigated the β -expansion of an algebraic number a in an algebraic base β . Using tools from Diophantine approximation it is so called by the *Diophantine exponent* of a and denoted by $Dio(a)$. later on, Ito-Sadahiro [11] discovered the representation of real numbers with an arbitrary negative base $-\beta < -1$, which is denoted by $(-\beta)$ -expansion and they have proven a lot of its algebraic properties. More details will be found in the next section.

In section (3) we will mainly focus on $(-\beta)$ -expansion, and prove that $Dio\left(d_{-\beta}(l_{\beta})\right)$ is bounded by $\frac{\log M(\beta)}{\log \beta} + (d-1)\frac{\log |a_d|+1}{\log \beta}$ such that β is an algebraic number.

Preliminaries

In this section, we start by giving several basic concepts which will be used in the rest of this paper.

Definition 2.1. $\mathbb{Q}[X]$ is the set of polynomials with coefficients in $\mathbb{Q}$ and $\mathbb{Z}[X]$ is the set of polynomials with coefficients in $\mathbb{Z}$.

Definition 2.2. A complex number $\alpha \in \mathbb{C}$ is algebraic if there exists a polynomial $f \in \mathbb{Q}[X]$ such that $f(\alpha) = 0$. Otherwise, α is transcendental.

Proposition 2.3. $\alpha \in \mathbb{C}$ is algebraic if and only if there exists a polynomial $f \in \mathbb{Z}[X]$ with integer coefficients such that $f(\alpha) = 0$.

Definition 2.4. Let $\alpha \in \mathbb{C}$ be algebraic. The degree of α , denoted as $\deg(\alpha)$, is the minimum degree of a polynomial $f \in \mathbb{Z}[X]$ such that $f(\alpha) = 0$.

Definition 2.5. [6] Let $P(x) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_0 = a_0 \prod_{i=1}^d (Z - \alpha_i)$ be a nonconstant polynomial with (at first) complex coefficients. Then the Mahler measure is defined to be $M(P) := \exp\left(\int_0^1 \log|P(e^{2\pi it})| dt\right)$, the geometric means $|P(z)|$ for z on the unit circle. However, let $M(\alpha)$ be the Mahler measure of the minimal polynomial P_α of an algebraic number α with d the degree of α , having conjugates $\alpha = \alpha_1, \alpha_2, \dots, \alpha_d$. If we consider a complete set of inequivalent valuations $|\cdot|_v$ of the field $\mathbb{Q}(\alpha)$, normalized so that, for $v|p$, $|\cdot|_v = |\cdot|_p$ on $\mathbb{Q}_p$. Here $\mathbb{Q}_p$ is the field of p -adic numbers, with the usual valuation $|\cdot|_p$. Then the Mahler measure in the form $M(\alpha)$ is

$$M(\alpha) = \prod_{all\ v} \max(1, |\alpha|_v^{d_v}).$$

The Diophantine approximation is the method of finding successively better rational approximations to irrational numbers, because $\mathbb{Q}$ is a dens set in $\mathbb{R}$ so any irrational number can be approximated arbitrarily and very close by rational numbers, i.e. Diophantine approximation is defined as the study of how well or quickly such approximations can be made. Let $v > 0$ and $n \in \mathbb{N}$, say that $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ is v -approximable if there are infinitely many $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$ such that:

$$|y_1 q_1 + y_n q_n + p| < \|q\|^{-v}$$

for some $p \in \mathbb{Z}$. the Diophantine exponent $Dio(a)$ of a is

$$Dio(a) = \sup\{v \mid y \text{ is } v\text{-approximable}\}.$$

Theorem 2.6. [4] Let $\beta > 1$ and $x \in [0, 1]$ be algebraic numbers such that $d_\beta(x) = 0$. $a_1 a_2 \dots$

Then, either the sequence $\mathbf{a} = (a_i)_{i \geq 1}$ is eventually periodic, or $Dio(\mathbf{a}) \leq \frac{\log(M(\beta))}{\log \beta}$, where $M(\beta)$ is the Mahler measure of β .

(- β) -negative expansion base

Definition 3.1. [11] Let $\beta > 1$ be a real number. A $(-\beta)$ -representation of a real number x is an expression of the form,

$$x = x_{-k}(-\beta)^k + x_{-k+1}(-\beta)^{k-1} + \dots + x_0 + \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \dots,$$

where $k \geq 0$ is a certain integer and $x_i > 0$ for $i \geq -k$. It is denoted by

$$x = (x_{-k} x_{-k+1} x_0 x_1 x_2 \dots)_{-\beta}$$

We denote by I_β the half-open interval $[l_\beta, r_\beta) = \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. The $(-\beta)$ -transformation T_β on I_β is defined by $T_{-\beta}(x) := -\beta x - \lfloor -\beta x - l_\beta \rfloor = \{-\beta x - l_\beta\} + l_\beta$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding a real number x and $\{x\} = x - \lfloor x \rfloor$.

Then, for each $x \in I_\beta = [l_\beta, r_\beta)$ we have a particular $(-\beta)$ -representation

$$x = (.x_1 x_2 \dots)_{-\beta},$$

$x_i = \lfloor -\beta T_{-\beta}^{i-1}(x) - l_\beta \rfloor$ for $i \geq 1$. We call this representation the $(-\beta)$ -expansion of x

Remark: The representation of the $(-\beta)$ -expansion of x can be written as an infinite word $d_{-\beta}(x) = x_1 x_2 x_3 \dots$. One can easily prove that the digits x_i , $i \geq 1$, take the values in the set $A = \{0, 1, 2, \dots, \lfloor \beta \rfloor\}$.

Mean results

As we mentioned, our concentration is to prove that the Diophantine exponent $Dio\left(d_{-\beta}(l_\beta)\right)$ is bounded by $\frac{\log M(\beta)}{\log \beta} + (d-1) \frac{\log |a_d| + 1}{\log \beta}$, where β is an algebraic number, and $M(\beta)$ is the Mahler measure of β . But before that, we need to prove the following lemmas, and finally the mean result will be in Theorem 4.3.

Lemma 4.1. (Classical theorem on symmetric polynomials)

Let $Q \in \mathbb{Z}[X]$ and $P(y_1, y_2, \dots, y_n) = Q(y_1), Q(y_2), \dots, Q(y_n)$. Then there exists a polynomial T of n variables with coefficients in $\mathbb{Z}$, such as

$$P(y_1, y_2, \dots, y_n) = T(\sigma_1, \sigma_2, \dots, \sigma_n) \quad \text{whrer:} \begin{cases} \sigma_1 = \sum_{i=1}^n y_i \\ \sigma_2 = \sum_{1 \leq i < j \leq n} y_i y_j \\ \sigma_2 = \sum_{1 \leq i < j < k \leq n} y_i y_j y_k \\ \vdots \\ \sigma_n = y_1, y_2, \dots, y_n \end{cases}$$

Moreover, we note that the total degree of T is less than or equal to the degree of Q (as a polynomial in y).

Lemma 4.2. Let β be an algebraic number with minimal polynomial

$$P_\beta(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0 \quad \text{where } a_i \in \mathbb{Z} \text{ for all } 0 \leq i \leq d.$$

Let $Q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$ where $b_i \in \mathbb{Z}$ for all $0 \leq i \leq n$, such that $Q(\beta) \neq 0$. Then, we have

$$|Q(\beta)| \geq \frac{\beta^n}{(H(Q))^{d-1} (n+1)^{d-1} (M(\beta))^n |a_d|^{n(d-1)+d}},$$

Where $H(Q)$ is the height defined by $H(Q) = \max_{0 \leq i \leq n} |b_i|$.

Proof: Let $\beta_0 = \beta, \dots, \beta_{d-1}$ be the conjugates of β and $P_\beta(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_0$ where $a_i \in \mathbb{Z}$ for all $0 \leq i \leq d$. The minimal polynomial of β , then is given by

$$a_d \beta^d = \sum_{i=0}^{d-1} -a_i \beta^i, \text{ therefore, for all } K \in \mathbb{N}^*;$$

$$a_d^k \beta^{d+k-1} = \sum_{i=0}^{d-1} \alpha_{k,i} \beta^i, \quad \alpha_{k,i} \in \mathbb{Z}.$$

This gives that for all $s' \geq s$:

$$a_d^{s'} \beta^s = \sum_{i=0}^{d-1} c_{k,i} \beta^i, \quad c_{k,i} \in \mathbb{Z}.$$

$$\text{So, } a_d^n Q(\beta) = \sum_{i=0}^{d-1} \theta_{k,i} \beta^i, \quad \theta_{k,i} \in \mathbb{Z}.$$

Therefore:

$$a_d^n Q(\beta_j) = \sum_{i=0}^{d-1} e_{k,i} \beta_j^i, \quad e_{k,i} \in \mathbb{Z} \quad \forall 0 \leq j \leq d-1.$$

Hence, we obtain $a_d^{nd} \prod_{j=0}^{d-1} Q(\beta_j) = \prod_{j=0}^{d-1} (\sum_{i=0}^{d-1} e_{k,i} \beta_j^i)$. By lemma 4.1, there exists the polynomial T of d variables with coefficients in $\mathbb{Z}$ such that

$$\prod_{j=0}^{d-1} (\sum_{i=0}^{d-1} e_{k,i} \beta_j^i) = T(\sigma_1, \sigma_2, \dots, \sigma_d),$$

Where

$$\sigma_j = \sum_{0 \leq k_1 < k_2 < \dots < k_j} \beta_{k_1} \beta_{k_2} \dots \beta_{k_j}, \quad 1 \leq j \leq d,$$

And the total degree of T is lower than or equal to d . which gives that

$$a_d^d a_d^{nd} \prod_{j=0}^{d-1} Q(\beta_j) \in \mathbb{Z}^*. \text{ Consequently, } |\prod_{j=0}^{d-1} Q(\beta_j)| \geq \frac{1}{|a_d|^{nd+d}}.$$

$$|Q(\beta_j)| < H(Q)(n+1) \sup(|\beta_j|^n, 1), \text{ so}$$

$$\prod_{j=1}^{d-1} Q(\beta_j) \leq H(Q)^{d-1} \prod_{j=1}^{d-1} (n+1) \sup(|\beta_j|^n, 1),$$

$$\leq \frac{(n+1)^{d-1} H(Q)^{d-1} (M(\beta))^n}{|a_d|^n \beta^n}.$$

Hence, we obtain

$$|Q(\beta)| \geq \frac{1}{\left| \prod_{j=1}^{d-1} Q(\beta_j) \right| |a_d|^{nd+d}} \geq \frac{\beta^n}{(H(Q))^{d-1} (n+1)^{d-1} (M(\beta))^n |a_d|^{n(d-1)+d}}.$$

■

Theorem 4.3. Let $\beta > 1$ be an algebraic number of algebraic degree ≥ 2 and $P(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$ its minimal polynomial, where $a_i \in \mathbb{Z}$ for all $0 \leq i \leq d$. If $d_{-\beta}(l_\beta) = a = (\epsilon_i)_{i \geq 1}$ is not eventually a periodic sequence, then

$$Dio(a) \leq \frac{\log M(\beta)}{\log \beta} + (d-1) \frac{\log |a_d| + 1}{\log \beta}.$$

Proof: Assuming that $U_n V_n^{w_n}$ is a prefix of a such that $|U_n| = k_n, |V_n| = t_n$ and $|V_n^{w_n}| = t_n w_n$, for all $n \geq 1$. If $V_n = \epsilon_{k_n+1} \epsilon_{k_n+2} \dots \epsilon_{k_n+t_n}$, then by the periodicity we define the sequence $(x_{k_n+j})_{j \geq 1}$ as following $x_{k_n+lt_n+i} = \epsilon_{k_n+i}$ for all $1 \leq i \leq t_n$, and $l \geq 0$.

Let

$$\begin{aligned} \lambda_{k_n} &= \sum_{i=1}^{+\infty} x_{k_n+i} (-\beta)^{-i} = \sum_{i=1}^{t_n} \epsilon_{k_n+i} (-\beta)^{-i} + (-\beta)^{-t_n} \lambda_{k_n}, \\ &= \frac{\epsilon_{k_n+1} (-\beta)^{t_n-1} + \dots + \epsilon_{k_n+t_n}}{(-\beta)^{t_n} - 1}. \end{aligned}$$

As $d_{-\beta}(l_\beta) = (\epsilon_i)_{i \geq 1}$, we have

$$\frac{-\beta}{\beta+1} = \frac{\epsilon_1}{(-\beta)} + \dots + \frac{\epsilon_{k_n}}{(-\beta)^{k_n}} + \frac{\epsilon_{k_n+1}}{(-\beta)^{k_n+1}} + \dots + \frac{\epsilon_{k_n+t_n}}{(-\beta)^{k_n+t_n}} + \dots$$

Where

$$\begin{aligned} \theta_{k_n} &= (-\beta)^{k_n+1} - (\beta+1) [\epsilon_1 (-\beta)^{k_n-1} + \dots + \epsilon_{k_n}] \\ &= (\beta+1) \left[\frac{\epsilon_{k_n+1}}{(-\beta)} + \dots + \frac{\epsilon_{k_n+t_n}}{(-\beta)^{t_n}} + \dots \right]. \end{aligned}$$

Let

$$Q_{k_n}(x) = ((-x)^{t_n} - 1) [(-x)^{k_n+1} - (x+1) (\epsilon_1 (-x)^{k_n-1} + \dots + \epsilon_{k_n}) - (x+1) (\epsilon_{k_n+1} (-x)^{t_n-1} + \dots + \epsilon_{k_n+t_n})]$$

A polynomial with integer coefficients of degree $k_n + t_n + 1$, Then we have

$$Q_{k_n}(\beta) = ((-\beta)^{t_n} - 1) (\theta_{k_n} - (\beta+1) \lambda_{k_n}).$$

Furthermore, $Q_{k_n}(\beta) \neq 0$ and $d_{-\beta}(l_\beta)$ are not eventually a periodic sequence. By using lemm4.2, we get

$$Q_{k_n}(\beta) \geq \frac{\beta^{k_n+t_n+1}}{[\beta]^{d-1} (t_n + k_n + 2)^{d-1} (M(\beta))^{k_n+t_n+1} |a_d|^{k_n+t_n+1(d-1)+d}} \quad (4.1)$$

On the other hand, we have:

$$\begin{aligned} |(\theta_{k_n} - (\beta+1) \lambda_{k_n})| &= \left| (\beta+1) \left[\sum_{i=1}^{+\infty} \frac{\epsilon_{k_n+i}}{(-\beta)^i} + \sum_{i=1}^{+\infty} x_{k_n+i} (-\beta)^i \right] \right|, \\ &\leq (\beta+1) \sum_{i=t_n w_n+1}^{+\infty} \frac{|\epsilon_{k_n+i} - x_{k_n+i}|}{\beta^i}, \\ &\leq (\beta+1) \frac{[\beta]}{\beta^{t_n w_n+1} \left(1 - \frac{1}{\beta}\right)} \\ &\leq \frac{(\beta+1)}{1 - \frac{1}{\beta} \beta^{-t_n w_n}}. \end{aligned}$$

Therefore

$$|Q_{k_n}(\beta)| \leq \frac{(\beta + 1)}{1 - \frac{1}{\beta}} \beta^{t_n - t_n w_n}. \quad (4.2)$$

Using equations (4.1) and (4.2), we get:

$$\beta^{k_n + t_n w_n + 1} \left(1 - \frac{1}{\beta}\right) \leq (\beta + 1)[\beta]^{d-1} (M(\beta))^{t_n + k_n + 1} |a_d|^{t_n + k_n + 1(d-1)+d} (t_n + k_n + 2)^{d-1} (k_n + t_n w_n + 1) \log \beta + \log \left(1 - \frac{1}{\beta}\right) \leq \log(\beta + 1) + (d - 1) \log[\beta] + (t_n + k_n + 1) \log(M(\beta)) + [(t_n + k_n + 1)(d - 1) + d] \log |a_d| + (d - 1) \log(t_n + k_n + 2)$$

If we divided by $[(k_n + t_n + 1) \log \beta]$, yields

$$\begin{aligned} & \frac{(k_n + t_n w_n + 1)}{(k_n + t_n + 1)} \\ & \leq \frac{1}{(k_n + t_n + 1)} \left[\frac{\log(\beta + 1) + (d - 1) \log[\beta] + d \log |a_d| - \log \left(1 - \frac{1}{\beta}\right)}{\log \beta} \right] + \frac{\log(M(\beta))}{\log \beta} + \frac{(d - 1) \log |a_d|}{\log \beta} + \frac{(d - 1)}{\log \beta}. \end{aligned}$$

Since $t_n w_n = |V_n^{w_n}| \rightarrow +\infty$ when $n \rightarrow +\infty$, then we have $k_n + t_n = |U_n V_n| \rightarrow +\infty$ when $n \rightarrow +\infty$, because in the otherwise the left-hand side $\frac{(k_n + t_n w_n + 1)}{(k_n + t_n + 1)}$ is unbounded, whereas in the right-hand side is bounded by a positive constant. We conclude that

$$\begin{aligned} Dio(a) &= \lim_{n \rightarrow +\infty} \sup \frac{|U_n V_n^{w_n}|}{|U_n V_n|} \\ &= \lim_{n \rightarrow +\infty} \sup \frac{k_n + t_n w_n}{k_n + t_n} \\ &\leq \frac{\log(M(\beta))}{\log \beta} + \frac{(d - 1) \log |a_d| + 1}{\log \beta}. \end{aligned}$$

■

The last theorem gives a family of transcendental numbers by using the following result.

Result: let $\beta > 1$ and $d_{-\beta}(l_\beta) = a = (\epsilon_i)_{i \geq 1}$ if $Dio(a) = +\infty$, then a is a transcendental number.

Corollary 4.4.

Let $\beta > 1$ be a real number such that the $(-\beta)$ -expansion of l_β is the form $d_{-\beta}(l_\beta) = 0.t_1 t_2 t_3 \dots$, with $t_i \in A_\beta = \{0, 1, 2, \dots, [\beta]\}$.

Assume that $d_{-\beta}(l_\beta)$ is an infinite sequence satisfying the following property: there exist two sequences $(m_n)_{n \geq 1}$ and $(s_n)_{n \geq 1}$ such that

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \dots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \dots$$

With $(s_n - m_n) \geq 2$, $a_i = p$ if $m_n < i < s_n$ for all $n \geq 1$. Then,

$$\lim_{n \rightarrow +\infty} \sup \frac{s_n}{m_n} \leq \frac{\log(M(\beta))}{\log \beta} + \frac{(d - 1) \log |a_d| + 1}{\log \beta} - 1.$$

Proof: Let U_n be the prefix of a of length m_n and let $V_n = P$. by assumption, $U_n V_n^{s_n}$ is a prefix of a and $\frac{|U_n V_n^{s_n}|}{|U_n V_n|} = \frac{m_n + s_n}{m_n + 1}$. It follows that

$$Dio(a) = \lim_{n \rightarrow +\infty} \sup \frac{|U_n V_n^{s_n}|}{|U_n V_n|} = 1 + \lim_{n \rightarrow +\infty} \sup \frac{s_n}{m_n},$$

From Theorem 3.1, we have

$$\lim_{n \rightarrow +\infty} \sup \left(\frac{s_n}{m_n} \right) \leq \frac{\log(M(\beta))}{\log \beta} + \frac{(d - 1) \log |a_d| + 1}{\log \beta} - 1.$$

References

1. A. Rényi Representations for real numbers and their ergodic properties, *Acta. Math. Acad. Sci. Hung.* 8 (1957), Pages 477-493.
2. B. Solomyak, Conjugates of beta-nombres and the zero free domain for a class of analytic functions, *London Math. Soc.* (1994), Pages 477-498.
3. B. Adamczewski and Y. Bugeaud, On the complexity of algebraic numbers I. Expansions in integer bases, *Annals of Mathematics* 165 (2007), Pages 547-565.
4. B. Adamczewski and Y. Bugeaud, *Dynamics for β -shifts and Diophantine approximation*, Cambridge University Press: 01 December 2007.
5. Ch. Frougny and A. Chiara Lai, Negative bases and automata, *Discrete Mathematics and Theoretical Computer Science* vol. 13:1, 2011, Pages 75-94.
6. C. SMYTH, The Mahler measure of Algebraic numbers: A Survey the, 2000 Mathematics Subject Classification. Primary 11R06, Pages 322-349.
7. F. Blanchard, β -expansions and symbolic dynamics, *Theoretical Computer Science*, Volume 65, Issue 2, (1989), Pages 131-141.
8. K. Mahler, Arithmetic properties of lacunary power series with integral coefficients, *The Journal of the Australian Mathematical Society*, Vol. V Part 1, (1965), Pages 56-64.
9. K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers, *Bull. London Mathematical Society*, (1980), Pages 269-278.
10. R. Güting, Approximation of algebraic numbers by algebraic numbers, *Michigan Math. J.* 8, (1961), Pages 149-159.
11. S. Ito and T. Sadahiro, Beta-expansions with negative bases, *Integers*. 9, (2009), Pages 239-259.
12. W. Parry, On the β -expansions of real numbers, *Acta Mathematica Academiae Scientiarum Hungarica*, (1960), Pages 401-416.
13. Z. Masáková and E. Pelantová, S. Ito-Sadahiro numbers vs. Parry numbers, *Acta ploytechnica*, 51, (2011), Pages 59-64.