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Exploring Prime Near-Rings and Semiprime Rings Under Skew and Generalized Skew Derivations

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Abstract: This paper investigates the structural properties of prime near-rings and prime rings under the action of skew derivations and generalized skew derivations. We examine how these mappings influence algebraic identities, commutativity conditions, and the behavior of ideals within such algebraic systems. By establishing new characterizations and commutativity criteria, the study extends classical results on prime rings to a broader framework that includes near-rings. Furthermore, we demonstrate how generalized skew derivations unify and extend several well-known derivation concepts, thereby providing a more flexible approach to analyzing algebraic structures. These findings contribute to the deeper understanding of prime near-rings and semiprime rings, offering new tools for further research in ring theory and its applications.

Keywords: Prime near-ring, semiprime ring, skew-derivation, 2-torsion free.

I. Introduction

The study of derivations and their generalizations has long been central to the development of modern algebra, particularly in the analysis of structural properties of rings and related algebraic systems Boua, A. (2012). Derivations provide powerful tools for exploring commutativity, ideal structure, and functional identities, and have been widely applied in both commutative and non-commutative settings. In recent years, attention has increasingly shifted toward skew derivations and generalized skew derivations, which extend the classical notion of derivations by incorporating endomorphisms and more flexible operator mappings, Rumah et al. (2023). These generalizations not only broaden the scope of classical results but also yield deeper insights into the internal symmetries and constraints of algebraic systems.

Prime rings and prime near-rings form a particularly important class of structures in this context. A prime ring, by definition, avoids trivial zero products of ideals, thereby exhibiting strong irreducibility properties, Golbasi, O. (2012). Near-rings, on the other hand, generalize rings by relaxing distributive conditions, making them suitable for modeling a wider range of algebraic phenomena, Tasiu et al. (2017). Investigating how skew derivations and generalized skew derivations act on such systems provides new perspectives on commutativity conditions, the behavior of ideals, and the preservation of primeness.

A substantial body of work has been devoted to understanding derivations on prime rings, with classical results characterizing when such operators force commutativity or influence ring identities. However, the extension of these results to near-rings remains relatively underexplored, despite the fact that near-rings naturally arise in areas such as automata theory, coding theory, and combinatorial design. By extending the framework of derivational mappings to prime near-rings, we aim to bridge this gap and establish results that not only generalize existing theorems but also highlight the structural nuances between rings and near-rings.

In this paper, we examine prime near-rings and prime rings under the action of skew derivations and generalized skew derivations. Our focus lies in analyzing how these mappings influence algebraic identities, commutativity criteria, and the structure of ideals within these systems. We establish new characterizations of primeness in the presence of such operators, thereby extending known results from ring theory to a more general near-ring framework. Furthermore, by demonstrating how generalized skew derivations unify and extend several known derivation concepts, we provide a more versatile approach to analyzing algebraic structures.

The findings presented here contribute to the deeper understanding of prime near-rings and prime rings, while also laying the groundwork for future investigations into operator-induced structures in non-commutative algebra. In particular, our results may have implications for applications where generalized algebraic frameworks play a role, including functional analysis, coding theory, and cryptographic systems.

II. Results

In this work, we explore the application of derivation and generalized derivation and established some theorems using skew derivation and generalized skew derivation. The primary objective is to extend the existing framework by introducing skew structures and demonstrating their efficacy in algebraic setting.

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Skew Derivation in Prime Near Rings

Some results on the commutativity of prime near-ring using skew derivation.

Theorem

Let N be a prime near-ring. If N admits a non-zero α-derivation d such that for all $x, y \in N$, $d([x,y]) = 0$, then N commutative.

Proof:

Suppose that,

$$d([x,y]) = 0 \text{ for all } x, y \in N \qquad (1)$$

We are to show that N is commutative.

Substitute xy for y in (1), we have

$$d([x,xy]) = d(x[x,y]) = d(x)[x,y] + \alpha(x)d[x,y], \text{for all } x, y \in N$$

but $d([x,y]) = 0$, we have

$$= d(x)[x,y] + \alpha(x).(0) \text{ for all } x, y \in N$$

$$= d(x)[x,y]$$

Since $d([x,xy]) = 0$, this implies

$$d([x,xy]) = d(x)[x,y]$$

that is

$$d(x)xy = d(x)yx, \text{for all } x, y \in N. \qquad (2)$$

Substitute yz for y in (2) we get,

$$d(x)x(yz) = d(x)(yz)x, \quad \text{for all } x, y, z \in N$$

implies

$$d(x)y(xz) = d(x)y(zx) \text{ for all } x, y, z \in N$$

$$d(x)N[x,z] = \{0\} \text{ for all } x, z \in N,$$

$$d(x)N[x,z] = 0\,,$$

by primeness of N we have,

$$d(x) = 0 \text{ or } [x,z] = 0 \text{ for all } x, z \in N$$

implies $d(x) = 0$ or $xz - zx = 0$ for all $x, z \in N$

which yield $d(x) = 0$ or $xz = zx$ for all $x, z \in N$

which conclude that N is commutative ring. Hence the prove.

Theorem

If N is any prime near-ring admitting a α-derivations d, with $\alpha(x) = x$, for all $x \in N$,where α is an identity automorphism such that for all $x, y \in N$ satisfying $d([x,y]) = [x,y]$ for all $x, y \in N$, then N is commutative.

Proof: Assume that,

$$d([x,y]) = [x,y] \text{ for all } x, y \in N \qquad (3)$$

Replacing y by xy in (3), and using the identity $[x,xy] = x[x,y]$, we get

$$x[x,y] = d(x[x,y]) \text{ for all } x, y \in N$$

and

$$d(x[x,y]) = d(x)[x,y] + \alpha(x)d([x,y]),$$

Since $\alpha(x) = x$, for all $x \in N$, we have

$$d(x[x,y]) = d(x)[x,y] + xd([x,y]),$$

then by using (3) in the above relations, we obtain

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$$x[x,y] = x[x,y] + d(x)[x,y].$$

This implies $d(x)[x,y] = 0$.

Hence $d(x)(xy - yx) = 0$ for all $x, y \in N$. (4)

Substituting yz for y in (4), we obtain

$$d(x)y(xz - zx) = 0 \text{ , for all } x, y, z \in N,$$

which leads to

$$d(x)N(xz - zx) = 0 \text{ for all } x, z \in N. \quad (5)$$

Since N is prime, Eq. (5) reduces to,

$$d(x) = 0 \text{ or } [x,z] = 0 \text{ for all } x, z \in N. \quad (6)$$

From (6) it follows that for each fixed $x \in N$ we have,

$$d(x) = 0 \text{ or } x \in Z(N) \quad (7)$$

and

$$xz - zx = 0 \text{ for all } x, z \in N,$$

implies

$$xz = zx \text{ for all } x, z \in N. \quad (8)$$

Hence by (8) we conclude that N is commutative.

Theorem

Let N be a prime near-ring. If N admits a nonzero α-derivation d with an identity automorphism α, such that $d([x,y]) = -[x,y]$ for all $x, y \in N$, then N is a commutative.

Proof:

Given that

$$d([x,y]) = -[x,y] \quad (9)$$

Replace y with xy in (9) we have,

$$d([x,xy]) = -[x,xy] \text{ for all } x, y \in N,$$

it follows that

$$d(x[x,y]) = -x[x,y] \text{ for all } x, y \in N. \quad (10)$$

Thus,

$$d(x[x,y]) = d(x)[x,y] + \alpha(x)d([x,y]) \text{ for all } x, y \in N.$$

But since $d([x,y]) = -[x,y]$ we have,

$$= d(x)[x,y] + \alpha(x)(-[x,y])$$
$$= d(x)[x,y] - \alpha(x)[x,y] \text{ for all } x, y \in N$$

From (10) we have,

$$d(x)[x,y] - \alpha(x)[x,y] = -x[x,y]$$

Since we have an identity automorphism which is $\alpha(x) = x$, for all $x \in N$, this implies

$$-x[x,y] = d(x)[x,y] - x[x,y], \text{for all } x, y \in N$$

and therefore

$$d(x)[x,y] = 0 \text{ for all } x, y \in N \quad (11)$$

and obtain $d(x)y(xz - zx) = 0$, which leads to $d(x)N(xz - zx) = 0$ for all $x, z \in N$.

Since N is prime, equation (11) reduces to,

$$d(x) = 0 \text{ or } [x,z] = 0 \text{ for all } x, z \in N.$$

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From (6) it follows that for each fixed $x \in N$ we have,

$$d(x) = 0 \text{ or } x \in Z(N)$$

But $x \in Z(N)$also implies that $d(x) \in Z(N)$ and it follows that

$$d(x) \in Z(N) \text{ for all } x \in N.$$

That is $d(N) \subset Z(N)$, we conclude that N is a commutative ring. This completes the proof of our theorem.

Skew Derivation in Semiprime Rings

In this section, we prove a result on skew derivation of semiprime rings using the central ideal.

Theorem

If N is a semiprime ring admitting a α-derivation d with an identity automorphism α, satisfying $d([x,y] = -[x,y]$ for all $x, y \in N$, then K is a central ideal.

Proof.

Suppose that

$$[x,y] = -d([x,y]) \text{ for all } x, y \in K. \qquad (12)$$

Now for all $x, y, z \in K$, we have

$$[x,y]z + d([x,y]z) = z[x,y] + d(z[x,y]),$$

which yields

$$[x,y]z + d([x,y])z + \alpha([x,y])d(z) = z[x,y] + d(z)[x,y] + \alpha(z)d([x,y]).$$

Using Eq. (12) we have,

$$-d([x,y])z + d([x,y])z + \alpha([x,y])d(z) = -zd([x,y]) + d(z)[x,y] + \alpha(z)d([x,y]).$$

Which implies

$$\alpha([x,y])d(z) = -zd([x,y]) + d(z)[x,y] + \alpha(z)d([x,y]).$$

Since α is an identity automorphism we conclude that,

$$[x,y]d(z) = d(z)[x,y] \text{ for all } x, y, z \in K \qquad (13)$$

This shows that $d(K)$ centralizes K, and it follows from (13) that $[x,y]$ is in the center of K for all $x, y \in K$. This follows that the ideal K is commutative. Hence K is in tile center of R. In the event that $d([x,y] = -[x,y]$ for all $x, y \in N$.

Skew Derivation in 2-Torsion free Prime Near-Rings.

Theorem

Let N be a 2-torsion free prime nearing. If N admits a nonzero α-derivation d with $\alpha(x) = x$, for all $x \in N$, such that $d(xy \pm yx) = [d(x), y]$ for all $x, y \in N$, then N is commutative.

Proof. From the hypothesis, we have

$$d(xy \pm yx) = [d(x), y], \text{for all } x, y \in N. \qquad (14)$$

expanding (14) we have,

$$d(xy) \pm d(yx) = d(x)y - yd(x).$$

By the definition we have,

$$d(x)y + \alpha(x)d(y) \pm d(y)x + \alpha(y)d(x) = d(x)y - yd(x)$$

Since we have an identity automorphism, implies

$$xd(y) \pm d(y)x = 0. \qquad (15)$$

By substitution xy for y in (15), we get

$$xd(x)y \pm d(x)yx = 0, \text{for all } x, y \in N \qquad (16)$$

Again, substitution yz for y in (16) and use it to obtain

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$$d(x)y[x,z] = 0, \text{for all } x,y,z \in N,$$

which leads to

$$d(x)N[x,z] = \{0\}, \text{for all } x,z \in N.$$

Replace x by $d(y)$ in the above relation we have

$$d^2(y)N[d(y),z] = \{0\}, \text{for all } y,z \in N.$$

From the primeness of N, we obtain that $d(y) \in Z(N)$ or $d^2(y) = 0$, for each $y \in N$, this assures that N is commutative. Hence the prove.

Generalized Skew Derivation in Near Rings.

in this work we established some new result using generalized skew derivation.

Theorem

Let N be a near-ring admitting nonzero generalized α-derivations F associated with a derivation d. Then:

(i). $\alpha(x)(F(y)z + \alpha(y)d(z)) = \alpha(x)F(y)z + \alpha(xy)d(z), for\ all\ x,y,z \in N,$

(ii). $x(yd(z) + F(y)z) = xyd(z) + xF(y)z, for\ all\ x,y,z \in N.$

Proof.

(i). For all $x,y,z \in N$, we have

$$F(x(yz)) = d(x)yz + \alpha(x)F(yz)$$
$$= d(x)yz + \alpha(x)(F(y)z + \alpha(y)d(z)). \quad (17)$$

Also,

$$F((xy)z) = F(xy)z + \alpha(xy)d(z)$$
$$= d(x)yz + \alpha(x)F(y)z + \alpha(xy)d(z) \quad (18)$$

Equating (17) and (18), we get

$$d(x)yz + \alpha(x)(F(y)z + \alpha(y)d(z)) = d(x)yz + \alpha(x)F(y)z + \alpha(xy)d(z)$$

implies,

$$\alpha(x)(F(y)z + \alpha(y)d(z)) = \alpha(x)F(y)z + \alpha(xy)d(z), \text{for all } x,y,z \in N,$$

which conclude the result.

(ii). For all $x,y,z \in N$,

$$F(x(yz)) = \alpha(x)F(yz) + d(x)yz = \alpha(x)(\alpha(y)d(z) + F(y)z) + d(x)yz \quad (19)$$

On the other hand,

$$F((xy)z) = \alpha(xy)d(z) + F(xy)z = \alpha(xy)d(z) + \alpha(x)F(y)z + d(x)yz. \quad (20)$$

Equating (19) and (20), we get

$$\alpha(x)(\alpha(y)d(z) + F(y)z) + d(x)yz. = \alpha(xy)d(z) + \alpha(x)F(y)z + d(x)yz.$$

Implies,

$$\alpha(x)(\alpha(y)d(z) + F(y)z) = \alpha(xy)d(z) + \alpha(x)F(y)z \text{ for all } x,y,z \in N,$$

which conclude the result.

III. Discussion

The results obtained in this study demonstrate that skew derivations and generalized skew derivations play a crucial role in shaping the structural properties of prime near-rings and semiprime rings. By extending classical commutativity conditions from prime rings to near-rings, we highlight the robustness of these operators in enforcing algebraic regularities across a broader class of systems. Specifically, our theorems show that the presence of nontrivial skew derivations or generalized skew derivations often forces a near-ring to exhibit commutative behavior. This suggests that such operators act as powerful constraints, revealing deep connections between derivational mappings and primeness conditions.

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Furthermore, the introduction of generalized skew derivations provides a unifying framework that encapsulates several well-known operator types. This not only simplifies the theoretical landscape but also strengthens the applicability of the results to different algebraic settings. In particular, our findings confirm and extend earlier results in the literature, such as those of Bell and Daif (1995), Brešar (1993), and Miyan et al. (2024), thereby situating the present work within a continuing trajectory of research on derivations in non-commutative algebra.

The significance of these results lies in their potential applications. Near-rings are increasingly relevant in areas such as coding theory, automata theory, and cryptography, where structural constraints imposed by derivations may lead to enhanced security and more efficient algebraic design. Thus, the theoretical contributions made here are not only relevant to pure ring theory but also open new pathways for applied investigations in computational and information sciences.

IV. Conclusion

In this paper, we established new results on prime near-rings, semiprime rings, and 2-torsion free prime near-rings under skew derivations and generalized skew derivations. The findings extend classical commutativity theorems to a more general framework, demonstrating that the existence of such operators frequently enforces commutativity or centralizing conditions. By introducing generalized skew derivations, we provided a broader and more flexible approach that unifies various derivation concepts, thereby enriching the theory of near-rings and their structural analysis.

The outcomes of this study contribute to the deeper understanding of operator-induced structures in algebra and create a platform for further exploration of derivational mappings in other generalized algebraic systems. Future work may investigate the interaction of skew derivations with additional algebraic constraints, as well as applications in areas such as cryptography, functional analysis, and combinatorial design, where the properties of near-rings and rings play a foundational role.

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