

b - Generalized Pre-Closed Sets in Topological Spaces

M. Andal¹, V. Thiripurasundari²

¹Assistant Professor in Department of Mathematics, P.S.R.R College of Engineering, Sivakasi, Affiliated to Anna University, Chennai

²Associate Professor in Department of Mathematics, Sri S. Ramasamy Naidu Memorial College, Sattur, Affiliated to Maduuri Kamaraj University, Madurai

DOI: <https://doi.org/10.51583/IJLTEMAS.2025.1410000114>

Abstract: In this paper we introduce a new class of b-generalized closed sets, b- generalized open sets in topological spaces, and study some of its basic properties.

Keywords: g - closed, $g\alpha$ - closed, αg - closed, gp - closed, gb - closed, bgp - closed, bgp - open.

I. Introduction

The study of generalized closed sets in topological spaces was initiated by Levine[3] in 1970. Andrijevic[1] introduced a new class of generalized open sets in a topological space, the so called b-open sets. Maki et.al[2] defined αg - closed set and $g\alpha$ - closed set in 1994. Balachandran and Rani[5] defined the notion of generalized pre-closed sets. A.A.Omari[4] and M.S.M.Noorani introduced and studied the concept of generalized b- closed sets in topological spaces.

Preliminaries

Definition 2.1

A topology on a set X is a collection τ of subsets of X having the following properties:

- $\emptyset$ and X are in τ .
- The union of the elements of any subcollection of τ is in τ .
- The intersection of the elements of any finite subcollection of τ is in τ . A set X for which a topology τ has been specified is called a topological space.

Definition 2.2

Let (X, τ) be any topological spaces. A subset S of X is said to be a closed set in (X, τ) if its complement in X, namely $X \setminus S$, is open in (X, τ) .

Definition 2.3[5]

A subset A of a topological space X is called a pre - open if $A \subseteq \text{int}(\text{cl}(A))$ and pre - closed if $\text{cl}(\text{int}(A)) \subseteq A$.

Definition 2.4[5]

The pre closure of a subset A of X, denoted by $\text{pcl}(A)$ is the intersection of all pre closed sets containing A. The pre interior of a subset A of X, denoted by $\text{pint}(A)$ is the union of all pre open sets contained in A.

Definition 2.5[6]

A subset A of a topological space X is called a α - open if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and α - closed if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

Definition 2.6[1]

A subset A of a topological space X is called a b - open if $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$ and b - closed if $\text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A)) \subseteq A$.

Definition 2.7[7]

A subset A of a topological space X is called a regular - open if $A = \text{int}(\text{cl}(A))$ and regular - closed if $\text{cl}(\text{int}(A)) = A$.

Definition 2.8[3]

A subset A of a topological space X is called a generalized closed (briefly, g - closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.9[6]

A subset A of a topological space X is called a generalized α - closed (briefly, $g\alpha$ - closed) if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is α - open in X.

Definition 2.10[6]

A subset A of a topological space X is called a α - generalized closed (briefly, αg - closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

Definition 2.11[2]

A subset A of a topological space X is called a generalized pre - closed (briefly, gp - closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

Definition 2.12[4]

A subset A of a topological space X is called a generalized b - closed (briefly, gb - closed) if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

3. b - Generalized Pre Closed sets.**Definition 3.1**

A subset A of a topological space X is called a b - generalized pre closed (briefly, bgp - closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is b - open in X .

The family of all bgp - closed sets in a topological space X is denoted by $bgpc(X)$.

Example 3.2

Let $X = \{a, b, c\}$ and topology $\tau = \{X, \emptyset, \{b\}, \{b, c\}\}$, then $bgpc(X) = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$.

Theorem 3.3

Every closed set is a bgp - closed set.

Proof: Let A be any closed set in X such that $A \subseteq U$ where U is b - open set in X . Since A is closed set and $pcl(A) \subseteq cl(A)$, $pcl(A) \subseteq U$ and U is b - open in X . Hence A is a bgp - closed set.

Remark 3.4

The converse of the theorem need not be true as seen from the following example.

Example 3.5

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{b\}, \{b, c\}\}$. Let $A = \{c\}$, then A is bgp - closed set but not a closed set.

Theorem 3.6

Every pre - closed set is a bgp - closed set.

Proof: Let A be a pre - closed set in X such that $A \subseteq U$ where U is b - open in X . Since A is pre - closed and $pcl(A) = A$, $pcl(A) \subseteq U$ and U is b - open in X . Hence A is a bgp - closed set.

Remark 3.7

The converse of the theorem need not be true as seen from the following example.

Example 3.8

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Let $A = \{a, c\}$, then A is bgp - closed set but not a pre closed set.

Theorem 3.9

Every regular - closed set is a bgp - closed set.

Proof: Let A be a regular closed set in X such that $A \subseteq U$ where U is b - open in X . Since A is regular closed set and $pcl(A) \subseteq rcl(A)$, $pcl(A) \subseteq U$. Hence A is a bgp - closed set.

Remark 3.10

The converse of the theorem need not be true as seen from the following example.

Example 3.11

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{b\}, \{b, c\}\}$. Let $A = \{a\}$, then A is bgp - closed set but not a regular closed set.

Theorem 3.12

Every α - closed set is a bgp - closed set.

Proof: Let A be a α - closed set in X such that $A \subseteq U$ where U is b - open in X . Since A is a α - closed set and $\text{pcl}(A) \subseteq \alpha\text{cl}(A)$, $\text{pcl}(A) \subseteq U$. Hence A is a bgp - closed set.

Remark 3.13

The converse of the theorem need not be true as seen from the following example.

Example 3.14

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Let $A = \{a, c\}$, then A is bgp - closed set but not a α - closed set.

Theorem 3.15

Every $g\alpha$ - closed set is a bgp - closed set.

Proof: Let A be a $g\alpha$ - closed set in X such that $A \subseteq U$ where U is b - open in X . Since A is a $g\alpha$ - closed set and $\text{pcl}(A) \subseteq \alpha\text{cl}(A)$, $\text{pcl}(A) \subseteq U$. Hence A is a bgp - closed set.

Remark 3.16

The converse of the theorem need not be true as seen from the following example.

Example 3.17

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Let $A = \{a, c\}$, then A is bgp - closed set but not a $g\alpha$ - closed set.

Theorem 3.18

Every gp - closed set is a bgp - closed set.

Proof: Let A be a gp - closed set in X such that $A \subseteq U$ where U is b - open in X . Since A is a gp - closed set and $\text{pcl}(A) \subseteq U$. Hence A is a bgp - closed set.

Remark 3.19

The converse of the theorem need not be true as seen from the following example.

Example 3.20

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a, b\}\}$. Let $A = \{a\}$, then A is bgp - closed set but not a gp - closed set.

Theorem 3.21

Every g - closed set is a bgp - closed set.

Proof: Let A be a g - closed set in X such that $A \subseteq U$ where U is b - open in X . Since A is a g - closed set and $\text{pcl}(A) \subseteq \text{cl}(A)$, $\text{pcl}(A) \subseteq U$. Hence A is a bgp - closed set.

Remark 3.22

The reverse implication of the above theorem need not be true as seen from the following example.

Example 3.23

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{b\}, \{b, c\}\}$. Let $A = \{c\}$, then A is bgp - closed set but not a g - closed set.

Theorem 3.24

Every αg - closed set is a bgp - closed set.

Proof: Let A be a g - closed set in X such that $A \subseteq U$ where U is b - open in X . Since A is a αg - closed set and $\text{pcl}(A) \subseteq \alpha\text{cl}(A)$, $\text{pcl}(A) \subseteq U$. Hence A is a bgp - closed set.

Remark 3.25

The converse of the above theorem need not be true as seen from the following example.

Example 3.26

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a, b\}\}$. Let $A = \{b\}$, then A is bgp - closed set but not a αg - closed set.

Remark 3.27

The Reverse implication of the above theorem need not be true summarized in the following implicit diagram.

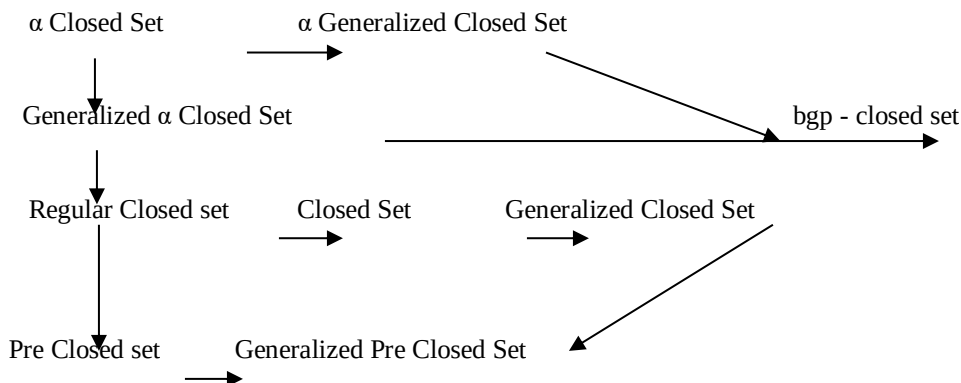


Diagram 1

Theorem 3.28

Let $B \subseteq X$ be a bgp - closed set, then $\text{pcl}(B) - B$ contains no non - empty b - closed set.

Proof: Let B be a bgp - closed set and M be a b - closed set in X such that $M \subseteq \text{pcl}(B) - B$, then $M \subseteq \text{pcl}(B)$ and $M \subseteq X - B$ implies $B \subseteq X - M$. Now, B is a bgp - closed set and $X - M$ is a b - open set containing B . It follows that $\text{pcl}(B) \subseteq X - M$ and thus $M \subseteq X - \text{pcl}(B)$. This implies $M \subseteq \text{pcl}(B) \cap (X - \text{pcl}(B)) = \emptyset$. Hence $M = \emptyset$. Therefore $\text{pcl}(B) - B$ contains no non - empty b - closed set.

Theorem 3.29

Let $A \subseteq X$ be a bgp - closed set, then A is pre - closed if and only if $\text{pcl}(A) - A$ is b - closed set.

Proof: Let $A \subseteq X$ be a bgp - closed set. Let A be a pre - closed set in X , then we have $\text{pcl}(A) - A = \emptyset$ which is b - closed set. Therefore $\text{pcl}(A) - A$ is b - closed set. Conversely, Assume that $\text{pcl}(A) - A$ is b - closed set. Now, A is bgp - closed set. Since $\text{pcl}(A) - A$ is a b - closed subset itself. By theorem 3.27, $\text{pcl}(A) - A = \emptyset$. This implies that $\text{pcl}(A) = A$ and so A is pre - closed. Therefore A is pre - closed.

Theorem 3.30

If $A \subseteq X$ is both b - open and bgp - closed, then A is pre - closed in X .

Proof: Let A be b - open and bgp - closed set in X , then $\text{pcl}(A) \subseteq A$. But $A \subseteq \text{pcl}(A)$ is always true. Therefore $\text{pcl}(A) = A$. Hence A is a pre - closed set.

Theorem 3.31

If $A \subseteq X$ is a bgp - closed set and $A \subseteq B \subseteq \text{pcl}(A)$, then B is bgp - closed in X .

Proof: Let U be a b - open set in X such that $B \subseteq U$, then $A \subseteq U$. Since A is a bgp - closed, then $\text{pcl}(A) \subseteq U$ and $A \subseteq B \subseteq \text{pcl}(A)$. Now, $\text{pcl}(B) \subseteq \text{pcl}(\text{pcl}(A)) = \text{pcl}(A) \subseteq U$. So, $\text{pcl}(B) \subseteq U$.

Remark 3.32

The converse of the above theorem need not be true as seen from the following example.

Example 3.33

Consider $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}\}$. Let $A = \{a\}$, $B = \{a, b\}$, then $A \subseteq B \subseteq \text{pcl}(A)$ and B is bgp - closed set in X but A is not a bgp - closed set in X .

Theorem 3.34

Let Y be an open subspace of a space X , and $A \subseteq Y$. If A is bgp - closed set in X , then A is bgp - closed set in Y .

Proof: Let U be a b - open set of Y such that $A \subseteq U$. Then $U = Y \cap H$ for some b - open set H of X . Since A is bgp - closed set in X , we have $\text{pcl}(A) \subseteq U$ and $\text{pcl}_Y(A) = Y \cap \text{pcl}(A) \subseteq Y \cap H = U$. Hence A is a bgp closed set in Y .

Remark 3.35

The converse of the above theorem need not be true as seen from the following example.

Example 3.36

Consider $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a, b\}\}$ and let $Y = \{a, b\}$ with the topology $\tau = \{\emptyset, X, \{a, b\}\}$. Let $A = \{a, b\}$, then $A \subseteq Y \subseteq X$ and A is bgp - closed set relative to Y but it is not bgp - closed relative to X .

Theorem 3.37

For a space X , the following statements are equivalent:

- (i) Every bgp closed set is gb - closed set.
- (ii) Every pre closed set is gb - closed.

Proof: (i)→(ii):

Given, every bgp - closed set is gb - closed. By above theorem, “every pre - closed set is bgp - closed set”. By our assumption, every pre closed set is gb - closed set.

(ii)→(i):

Given, every pre - closed set is gb - closed set. Let A be a bgp - closed set in X such that $A \subseteq U$ where U is b - open in X , then $\text{pcl}(A) \subseteq U$. Since $\text{pcl}(A)$ is pre - closed set, then by (ii), $\text{pcl}(A)$ is gb - closed set. Therefore $\text{cl}(A) \subseteq \text{cl}(\text{pcl}(A)) \subseteq U$. That is, $\text{cl}(A) \subseteq U$. Therefore A is a gb - closed set. Hence every bgp closed set is gb - closed set.

4. b - Generalized Pre Open Sets

Definition 4.1

A subset A of a topological space X is called a b - generalized pre open (briefly, bgp - open) set if A^c is bgp - closed.

Theorem 4.2

A set $A \subseteq X$ is bgp - open if and only if $U \subseteq \text{pint}(A)$ whenever U is b - closed and $U \subseteq A$.

Proof: Let A be a bgp - open set. Suppose that $U \subseteq A$ where U is b - closed set. Then $X - A$ is a bgp - closed set contained in the b-open set $X - U$, $\text{pcl}(X - A) \subseteq X - U$. Since $\text{pcl}(X - A) = X - \text{pint}(A)$, then $X - \text{pint}(A) \subseteq X - U$. That is $U \subseteq \text{pint}(A)$. Conversely, let $U \subseteq \text{pint}(A)$ whenever $U \subseteq A$ and U is b - closed set, then $X - \text{pint}(A) \subseteq X - U$. That is $\text{pcl}(X - A) \subseteq X - U$. This implies $X - A$ is bgp - closed set and A is bgp - open in X .

Theorem 4.3

If A is bgp - open set and B is any set in X such that $\text{pint}(A) \subseteq B \subseteq A$, then B is bgp - open set in X .

Proof: Let A is bgp - open and B is any set in X such that $\text{pint}(A) \subseteq B \subseteq A$. It follows from the definition 4.1 and theorem 3.22. Hence B is bgp - open set in X .

Theorem 4.4

If a set $A \subseteq X$ is bgp - closed set, then $\text{pcl}(A) - A$ is bgp-open set in X .

Proof: Suppose that A is bgp-closed set and M is a b-closed set such that $M \subseteq \text{pcl}(A) - A$. By theorem 3.19, $M = \emptyset$. Hence $M \subseteq \text{pint}(\text{pcl}(A) - A)$. Therefore by theorem 4.2, $\text{pcl}(A) - A$ is bgp - open set.

References

1. D. Andrijevic, On b-open sets, Mat.Vesnik 48(1996), 59-64.
2. H. Maki, J.Umehara and T. Noiri, Every topological spaces in pre- $T_{1/2}$, Mem.Fac.Sci.Kochi Univ.Ser.A.Math, 17(1996), 33-42
3. N. Levine, Generalized closed sets in topology, Amer.Math. Monthly 70 (1963), 36-41.
4. A.A.Omari and M.S.M.Noorani, On Generalized b-closed Sets, Bull.Malays. Math. Sci.Soc(2) 32 (2009), 19-30.
5. A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On Pre-Continuous and Pre Continuous mappings, Proc.Math.And Phys. Soc. Egypt. 53(1982), 47-53.
6. H. Maki, J.Umehara and K. Balachandran, Associated topologies of generalized α -closed sets and α -generalized closed sets, Mem.Fac.Sci.Kochi Univ. Ser. A Math. 15(1994), 51-63.
7. M.Stone, Applications of the theory of Boolean rings to general topology, Trans.Amer.Math.Soc, 41(1937), 374-481