

A Study on $\hat{\mu}\beta$ Connectedness in Bitopological Spaces

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DOI : <https://doi.org/10.51583/IJLTEMAS.2025.1411000086>

Received: 01 December 2025; Accepted: 06 December 2025; Published: 18 December 2025

ABSTRACT

The objective of this paper is to study a special case of connectedness in bitopological spaces by considering $\tau_1\tau_2 \hat{\mu}\beta$ open sets and examining their relationships with $\tau_1\tau_2 \square\square$ connected space and $\tau_1\tau_2$ pre-connected space.

Key words: $\tau_1\tau_2 \hat{\mu}\beta$ open set , $\tau_1\tau_2 \hat{\mu}\beta$ connected space .

INTRODUCTION

The study of bitopological spaces was initiated by Kelly, J.C [5] . A triple (X, τ_1, τ_2) is called bitopological space if (X, τ_1) and (X, τ_2) are two topological spaces . In 1997, Kumar Sampath S [6] introduced the concept of $\tau_1\tau_2$ - $\square\square$ -open sets in bitopological spaces. In 1981, Bose S [1] introduced the notion of $\tau_1\tau_2$ - semi - open sets in bitopological spaces . In 1992, Kar A [4] have introduced the notion of $\tau_1\tau_2$ - pre- open sets in bitopological spaces . In 2012 , H . I Al-Rubaye Qaye [2] introduced the notion of $\tau_1\tau_2$ - semi - $\square\square$ -open sets in bitopological spaces . In this paper, we study a special case of $\hat{\mu}\beta$ connectedness in bitopological spaces, and we prove several results by comparing them with similar cases in topological spaces.

Preliminaries

Throughout the paper, spaces always mean a bitopological spaces , the closure and the interior of any subset A of X with respect to τ_i , will be denoted by $\tau_i cl A$, and $\tau_i int A$ respectively, for

$i \in \{1, 2\}$.

Definition 2.1 :

Let (X, τ_1, τ_2) be a bitopological space , $A \subseteq X$, A is said to be :

- (i) $\tau_1\tau_2$ pre-open set [4] if $A \subseteq \tau_1 int(\tau_2 cl(A))$.
- (ii) $\tau_1\tau_2 \hat{\mu}\beta$ closed set [1] if $\tau_2 \hat{\mu} cl(A) \subseteq U$, whenever $A \subseteq U$, U is β open in τ_1 .
- (iii) $\tau_1\tau_2 \square\square$ open set [6] if $A \subseteq \tau_1 int(\tau_2 cl(\tau_1 int(A)))$.

Remark 2.2 :

The family of $\tau_1\tau_2$ pre-open (resp. $\tau_1\tau_2 \hat{\mu}\beta$ open) sets of X is denoted by

$\tau_1\tau_2 PO(X)$ (resp. $\tau_1\tau_2 \hat{\mu}\beta O(X)$).

Example 2.3 :

Let $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$, and $\tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}\}$ is a bitopological space . The family of all $\tau_1\tau_2 \hat{\mu}\beta$ open sets of X is : $\tau_1\tau_2 \hat{\mu}\beta O(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Remark 2.4 :

It is clear by definition that in any bitopological space the following hold :

- (i) Every τ_1 open set is $\tau_1\tau_2\hat{\mu}\beta$ open , $\tau_1\tau_2$ semi -open , $\tau_1\tau_2\Box\Box$ open set .
- (ii) Every $\tau_1\tau_2\Box\Box$ -open set is $\tau_1\tau_2$ pre-open , $\tau_1\tau_2$ semi open set .
- (iii) The concept of $\tau_1\tau_2$ pre-open and $\tau_1\tau_2$ semi open sets are independent .

Proposition 2.5 :

A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2\Box\Box$ open set if and only if there exists an τ_1 open set $\Box U$, such that $U \Box\Box A \Box \tau_1\text{int}(\tau_2cl(U))$.

Proof :

This follows directly from the definition (2.1) (iii).

Proposition 2.6 : [7]

A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$ semi open set if and only if there exists an τ_1 open set $\Box U$, such that $U \Box\Box A \Box\Box \tau_2cl(U)$.

Proposition 2.7 : [3]

A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$ pre open set if and only if there exists an τ_1 open set $\Box U$, such that $A\Box U \Box\Box \tau_2cl(A)$.

Theorem 2.8 :

A subset A of a bitopological space (X, τ_1, τ_2) is an $\tau_1\tau_2\Box\Box$ open set if and only if A is $\tau_1\tau_2$ semi open set and $\tau_1\tau_2$ pre open set .

Proof :

Follows from definition (2.1) and remark (2.5) .

Definition 2.9 : [4,6]

Let (X, τ_1, τ_2) be a bitopological space and $A\Box\Box X$, the intersection of all $\tau_1\tau_2\Box\Box$ closed

(resp . $\tau_1\tau_2$ pre closed) sets containing A is called $\tau_1\tau_2\Box\Box$ closure (resp . $\tau_1\tau_2$ pre closure) of A , and is denoted by $\tau_1\tau_2\Box cl(A)$ (resp . $\tau_1\tau_2\text{pcl}(A)$) ;

i.e $\tau_1\tau_2\Box cl(A) \Box\Box\Box \{B \Box\Box X : B \text{ is } \tau_1\tau_2\Box\Box \text{ closed set , } A\Box\Box X\}$ and

$\tau_1\tau_2\text{pcl}(A) \Box\Box\Box \{B \Box\Box X : B \text{ is } \tau_1\tau_2\text{pre closed set , } A\Box\Box X\}$.

Remark 2.10 : [2]

- (i) Every τ_1 open set $\Box$ is $\tau_1\tau_2\hat{\mu}\beta\Box$ -open set, but the converse need not be true.
- (ii) If every τ_1 open set $\Box$ is $\tau_1\Box$ closed $\Box$ and every nowhere $\tau_1\Box$ dense set $\Box$ is $\tau_1\Box$ closed $\Box$ in any

bitopological space, then every $\tau_1\tau_2\hat{\mu}\beta$ open set is an τ_1 open set.

Remark 2.11 : [2]

(i) Every $\tau_1\tau_2$ open set is $\tau_1\tau_2\hat{\mu}\beta$ open set, but the converse is not true in general.

(ii) If every τ_1 open set is τ_1 closed set in any bitopological space, then every $\tau_1\tau_2\hat{\mu}\beta$ open set is an $\tau_1\tau_2$ open set.

Remark 2.12 : [2]

The concepts of $\tau_1\tau_2\hat{\mu}\beta$ -open and $\tau_1\tau_2$ pre open sets are independent, as the following example.

Example 2.13:

In example (2.3), $\{b, c\}$ is a $\tau_1\tau_2$ -pre-open set but not $\tau_1\tau_2$ -semi-open set.

Remark 2.14 : [2]

(i) It is clear that every $\tau_1\tau_2$ semi open and $\tau_1\tau_2$ pre open subsets of any bitopological space is

$\tau_1\tau_2\hat{\mu}\beta$ open set

(ii) An $\tau_1\tau_2\hat{\mu}\beta$ open set in any bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$ pre open set if every τ_1 open subset of X is τ_1 closed set (from remark (2.17) (ii) and remark (2.5) (iii)).

Definition 2.15 : [2]

The complement of $\tau_1\tau_2\hat{\mu}\beta$ open set is called $\tau_1\tau_2\hat{\mu}\beta$ closed set. Then family of all $\tau_1\tau_2\hat{\mu}\beta$ closed sets of X is denoted by $\tau_1\tau_2\hat{\mu}\beta C(X)$.

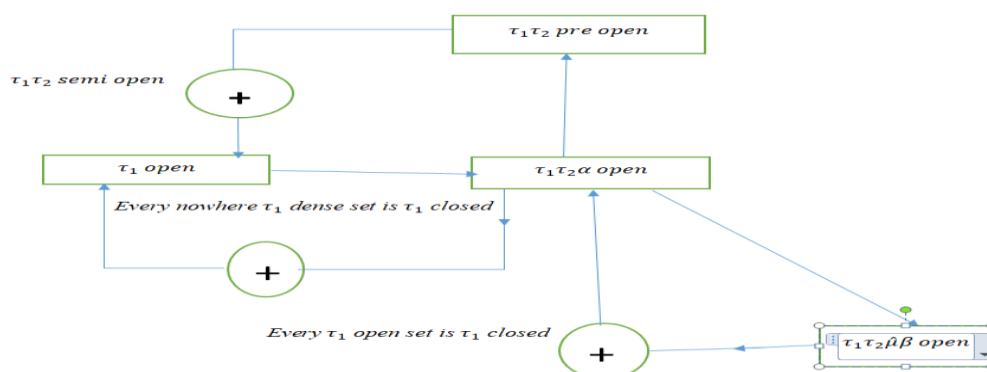
Definition 2.16 : [2]

Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$, the intersection of all $\tau_1\tau_2\hat{\mu}\beta$ closed sets containing A is called $\tau_1\tau_2\hat{\mu}\beta$ closure of A , and is denoted by $\tau_1\tau_2\hat{\mu}\beta cl(A)$;

i.e. $\tau_1\tau_2\hat{\mu}\beta cl(A) = \bigcap \{B \subseteq X : B \text{ is } \tau_1\tau_2\hat{\mu}\beta \text{ closed set, } A \subseteq B\}$.

Remark 2.17 : [2]

The following diagram shows the relations among the different types of weakly open sets that were studied in this section:



$\tau_1\tau_2\hat{\mu}\beta$ Connectedness in Bitopological Spaces :

In this section the notion of $\tau_1\tau_2\hat{\mu}\beta$ connected space is introduced in bitopological spaces and their relationships with $\tau_1\tau_2\Box\Box$ connected space and $\tau_1\tau_2$ pre connected space are studied.

Definition 3.1 :

Let $(X, \tau_1, \tau_2)\Box\Box$ be a bitopological space, two non-empty subsets A and B of X are said to be $\tau_1\tau_2\hat{\mu}\beta$ separated if $\Box A \Box \tau_1\tau_2\hat{\mu}\beta cl(B) \Box\Box\Box$ and $\tau_1\tau_2\hat{\mu}\beta cl(A) \Box\Box B \Box\Box\Box$.

Definition 3.2 :

A bitopological space $(X, \tau_1, \tau_2)\Box$ is called $\tau_1\tau_2\hat{\mu}\beta$ connected if it is not the union of two non empty $\tau_1\tau_2\hat{\mu}\beta$ separated $\tau_1\tau_2\hat{\mu}\beta$ opensets .

A subset $B \Box\Box X$ is $\tau_1\tau_2\hat{\mu}\beta$ connected if it is $\tau_1\tau_2\hat{\mu}\beta$ connected as a subspace of X .

An $\tau_1\tau_2\hat{\mu}\beta$ disconnection of X is a pair of complement, non-empty, $\tau_1\tau_2\hat{\mu}\beta$ open $\tau_1\tau_2\hat{\mu}\beta$ closed subsets.

Remark 3.3 :

The only $\tau_1\tau_2\hat{\mu}\beta$ open $\tau_1\tau_2\hat{\mu}\beta$ closed subsets in $\tau_1\tau_2\hat{\mu}\beta$ connected space X are X and $\Box\Box$.

Remark 3.4 :

Every $\tau_1\tau_2\hat{\mu}\beta$ connected space is τ_1 connected, but the converse is not true.

Proof :

Suppose that X is not τ_1 connected, then there exist an $\Box$ two non-empty A, B are $\tau_1\Box$ open such that $\Box A \Box\Box B \Box\Box\Box$ and $A \Box\Box B \Box\Box X$. we have A, B are $\tau_1\tau_2\hat{\mu}\beta$ open sets, $A \Box\Box B \Box\Box X$ and

$A \Box\Box B \Box\Box\Box$, hence X is not $\tau_1\tau_2\hat{\mu}\beta$ connected which is a contradiction. Thus, X is τ_1 connected.

But the converse is not true as in the following example.

Example 3.5 :

Let $X \Box \{a, b, c, d\}$, $\Box\Box \Box\Box\Box X$, $\Box\Box\Box a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$, and $\Box\Box\Box\Box\Box X, \Box\Box\Box a\}, \{c\}, \{a, c\}, \{a, b, c\}\Box$ is a bitopological space. The family of all $\tau_1\tau_2\hat{\mu}\beta$ open sets of X is : $\tau_1\tau_2\hat{\mu}\beta O(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Then X is $\Box\Box$ connected space, but X is not $\tau_1\tau_2\hat{\mu}\beta$ connected.

Remark 3.6 :

If every τ_1 open set $\Box\Box$ is τ_1 closed $\Box$ and every nowhere $\tau_1\Box$ dense set $\Box$ is $\tau_1\Box$ closed $\Box$ in any bitopological space, then every τ_1 connected space is $\tau_1\tau_2\hat{\mu}\beta$ connected .

Proof :

Follows from remark (2.16) (ii) .

Definition 3.7 :

A function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2\hat{\mu}\beta$ open if for each τ_1 open set U of X , $f(U)$ is $\tau_1\tau_2\hat{\mu}\beta$ open in Y .

Definition 3.8 :

A function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2\hat{\mu}\beta$ continuous if and only if the inverse image of each τ_1 open subset of Y is $\tau_1\tau_2\hat{\mu}\beta$ open subset of X .

Proposition 3.9 :

Every τ_1 continuous function is $\tau_1\tau_2\hat{\mu}\beta$ continuous.

Proof :

Follows from remark (2.16) (i) .

Definition 3.10 :

A function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2\hat{\mu}\beta$ irresolute if and only if the inverse image of each $\tau_1\tau_2\hat{\mu}\beta$ open subset of Y is $\tau_1\tau_2\hat{\mu}\beta$ open subset of X .

Proposition 3.11 :

Every $\tau_1\tau_2\hat{\mu}\beta$ irresolute function is $\tau_1\tau_2\hat{\mu}\beta$ continuous.

Proof :

Let A be any σ_1 open set in Y . Then we have A is an open set in $\tau_1\tau_2\hat{\mu}\beta$ open in Y [from remark (2.16) (i)] . Since f is $\tau_1\tau_2\hat{\mu}\beta$ irresolute function, then $f^{-1}(A)$ is $\tau_1\tau_2\hat{\mu}\beta$ open set in X . Therefore f is $\tau_1\tau_2\hat{\mu}\beta$ continuous. $\square$

Proposition 3.12 :

Let $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ be two bitopological spaces . If X is $\tau_1\tau_2\hat{\mu}\beta$ connected and f is $\tau_1\tau_2\hat{\mu}\beta$ continuous function from (X, τ_1, τ_2) onto (X, σ_1, σ_2) , then Y is τ_1 connected.

Proof :

Suppose that A is an τ_1 open τ_1 closed subset of Y , then $f^{-1}(A)$ is $\tau_1\tau_2\hat{\mu}\beta$ open $\tau_1\tau_2\hat{\mu}\beta$ closed in X . Hence $f^{-1}(A)$ is $\square$ or X , but X is $\tau_1\tau_2\hat{\mu}\beta$ connected . So A is $\square$ or Y . Hence Y is τ_1 connected.

Proposition 3.13 :

An $\tau_1\tau_2\hat{\mu}\beta$ irresolute image of any $\tau_1\tau_2\hat{\mu}\beta$ connected bitopological space is $\tau_1\tau_2\hat{\mu}\beta$ connected.

Proof :

Follows directly from proposition (3.12) .

Definition 3.14 :

Let (X, τ_1, τ_2) be a bitopological space , two non-empty subsets A and B of X are said to

be $\tau_1\tau_2$ separated (resp. $\tau_1\tau_2$ pre separated) if $A \cap \tau_1\tau_2 cl(B) = \square$ (resp. $A \cap \tau_1\tau_2 pcl(B) = \square$)

and $\tau_1\tau_2 \square cl(A) \square \square B \square \square \square \square$ (resp. $\tau_1\tau_2 \square pcl(A) \square \square B \square \square \square \square$).

Definition 3.15 :

A bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2 \square \square$ connected (resp. $\tau_1\tau_2$ pre connected) space if it is not the union of two non-empty $\tau_1\tau_2 \square \square$ separated (resp. $\tau_1\tau_2$ pre separated) $\tau_1\tau_2 \square \square$ open (resp. $\tau_1\tau_2$ pre open) sets.

Proposition 3.16 :

Every $\tau_1\tau_2$ pre connected space is $\tau_1\tau_2 \square \square$ connected.

Proof :

Follows from remark (2.5) (ii).

Proposition 3.17 :

If every $\tau_1\tau_2$ pre open set in a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$ semi open set , then X is $\tau_1\tau_2$ pre connected space , whenever it is an $\tau_1\tau_2 \square \square$ connected space .

Proof :

Follows from theorem (2.9).

Proposition 3.18 :

Every $\tau_1\tau_2$ pre connected space is τ_1 connected.

Proof :

Follows from remark (2.5) (i).

Remark 3.19 :

Every $\tau_1\tau_2\hat{\mu}\beta$ connected space is $\tau_1\tau_2 \square \square$ connected.

Proof :

Follows from remark (2.17) (i).

Proposition 3.20 :

In a bitopological space (X, τ_1, τ_2) if every $\tau_1 \square$ open subset $\square$ of X is $\tau_1 \square$ closed $\square$ set , then X is $\tau_1\tau_2\hat{\mu}\beta$ connected space , whenever it is an $\tau_1\tau_2 \square \square$ connected space.

Proof : Follows from remark (2.17) (ii).

Remark 3.21 :

The concepts of $\tau_1\tau_2$ pre connected space and $\tau_1\tau_2\hat{\mu}\beta$ connected space are independent.

