

Fermatean Fuzzy Maximal Subgroup and its Characterizations

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ABSTRACT

In this article, we investigate the concept of fermatean fuzzy characteristic subgroup of a group and fermatean fuzzy normal subgroup. The level subset of fermatean fuzzy subgroup and its properties also defined. Finally, the homomorphic image of fermatean fuzzy subgroup is also discussed.

Keywords: fuzzy set, fermatean fuzzy set, subgroup, characteristic, level subset, homomorphism.

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INTRODUCTION

The concept of fuzzy set is first introduced by L.A.Zadeh in 1965 [23]. After the tremendous development intuitionistic fuzzy set [4], pythagoren fuzzy set [22], (3,2) fuzzy set[15], picture fuzzy set [10], Neutrosophic fuzzy set [19] was studied. Recently fermatean fuzzy set [18] was studied by Yager and its application is briefly explained by [18]. Due to the inadequacy of FST in the sense that it admits only the membership grade, intuitionistic fuzzy sets (IFSs) was introduced [5, 6] by incorporating membership and non-membership grades with the chance for hesitation margin. Various properties of IFSs were presented [4,5] and IFSs have been applied in real-life problems [4, 5, 6]. Consequently, Biswas [8, 9] introduced intuitionistic fuzzy subgroups (IFSGs) based on IFSs, Ahn et.al. [2] deliberated on sublattice of lattice of IFSGs of a group, some properties of IFSGs were discoursed in [1]. In addition, Yuan et.al. [21] shared some light on the description of IFSGs, Bal et.al. [7] presented a brief note of kernel subgroups on IFGs and derived some properties of IFGs. Senapati.T [18] introduced fermatean fuzzy sets in 2020, an extension of intuitionistic fuzzy set (IFSs) and pythagorean fuzzy set (PFSs). AVs and NAVs of fermatean fuzzy sets reveal their dependence on greater powers with sum of cubes is less than 1. Ibrahim.H.Z introduced and applied n, m-rung orthopair fuzzy sets in MCDM.. The concepts of fuzzy modules and fuzzy sub-modules was introduced by Nagotia and Ralescu [16] in 1975. The concept of essential fuzzy modules was introduced by Hadi [13] in 2000. Using this idea. Abbas established the concept of essential fuzzy sub-modules and uniform fuzzy modules in 2012. In this article, we investigate the concept of fermatean fuzzy characteristic subgroup of a group and fermatean fuzzy normal subgroup. The level subset of fermatean fuzzy subgroup and its properties also defined. Finally, the homomorphic image of fermatean fuzzy subgroup is also discussed.

Preliminaries:

Throughout the article, the symbols S and G represent a non-empty set and a group respectively.

Definition-2.1(L.A.Zadeh): A fuzzy sub collection A of S is of the form

$A = \{(x, J_A(x))/x \in S\}$ where $J_A: X \rightarrow [0,1]$ is the membership grade of $x \in X$.

Definition-2.2(A.Rosenfeld): A fuzzy sub collection A of G is a fuzzy subgroup of G if

- (i) $J_A(xy) \geq \min\{J_A(x), J_A(y)\}$
- (ii) $J_A(x^{-1}) = J_A(x), \forall x, y \in G$.

where 'min' is a minimum operation.

Furthermore, $J_A(e) = J_A(xx^{-1}) \geq \min\{J_A(x), J_A(x)\} = J_A(x), \forall x \in G$, where e is the unit element of G. Then $J_A(e)$ is the upper bound or tip of A.

Definition-2.3(K.T. Atanassov):

An intuitionistic fuzzy set A of S is of the form $A = \{(x, J_A(x), K_A(x))/x \in S\}$ where $J_A: X \rightarrow [0,1]$ and $K_A: X \rightarrow [0,1]$ are the membership and non-membership grades of $x \in X$ and $0 \leq J_A(x) + K_A(x) \leq 1$.

Definition-2.4(Yager's): A fermatean fuzzy set A of S is of the form $A = \{(x, J_A(x), K_A(x))/x \in S\}$ where $J_A: X \rightarrow [0,1]$ and $K_A: X \rightarrow [0,1]$ are the membership and non-membership grades of $x \in X$ and $0 \leq J_A^3(x) + K_A^3(x) \leq 1$.

Definition-2.5: Let α_1 and α_2 be fermatean fuzzy sets of S. Then

- (i) $\alpha_1 = \alpha_2 \Leftrightarrow J_{\alpha_1}^3(x) = J_{\alpha_2}^3(x)$ and $K_{\alpha_1}^3(x) = K_{\alpha_2}^3(x), \forall x \in S$.
- (ii) $\alpha_1 \subseteq \alpha_2 \Leftrightarrow J_{\alpha_1}^3(x) \leq J_{\alpha_2}^3(x)$ and $K_{\alpha_1}^3(x) \geq K_{\alpha_2}^3(x), \forall x \in S$.
- (iii) $\alpha_1 \cap \alpha_2 = \{(x, \min\{J_{\alpha_1}^3(x), J_{\alpha_2}^3(x)\}, \max\{K_{\alpha_1}^3(x), K_{\alpha_2}^3(x)\})/x \in S\}$
- (iv) $\alpha_1 \cup \alpha_2 = \{(x, \max\{J_{\alpha_1}^3(x), J_{\alpha_2}^3(x)\}, \min\{K_{\alpha_1}^3(x), K_{\alpha_2}^3(x)\})/x \in S\}$.

Definition-2.6: Let $A = (J_A, K_A)$ be a fermatean fuzzy set on a group X. Then A is a fermatean fuzzy subgroup of X if

- (i) $J_A^3(x * y) \geq \min\{J_A^3(x), J_A^3(y)\}$ and $K_A^3(x * y) \leq \max\{K_A^3(x), K_A^3(y)\}$ for all $x, y \in X$.
- (ii) $J_A^3(x^{-1}) \geq J_A^3(x)$ and $K_A^3(x^{-1}) \leq K_A^3(x)$ for all $x \in X$.

Definition-2.7: If α_1 and α_2 are fermatean fuzzy subgroups of a group G, then α_1 is a fermatean fuzzy subgroup of α_2 if $\alpha_1 \subseteq \alpha_2$, and α_1 is a proper fermatean fuzzy subgroup of α_2 if $\alpha_1 \subset \alpha_2$, which gives α_1 is strictly contained in α_2 .

Definition-2.8: Let α_1 be a fermatean fuzzy subgroup of G. Then, the level set of α_1 is denoted by α_1^* and defined by $\alpha_1^* = \{x \in G/J_{\alpha_1}^3(x) \geq 0 \text{ and } K_{\alpha_1}^3(x) \leq 0\}$. Also α_1^* is a subgroup of G.

Definition-2.9: Suppose α_1 be a fermatean fuzzy subgroup of G. Then the strong $[\lambda, \mu]$ -cuts of α_1 is denoted by $\alpha_{1[\lambda, \mu]}$ and defined by $\alpha_{1[\lambda, \mu]} = \{x \in G/J_{\alpha_1}^3(x) \geq \lambda \text{ and } K_{\alpha_1}^3(x) \leq \mu\}$. Also it is a subgroup of G, where $\lambda, \mu \in [0, 1]$.

Similarly, the weak (λ, μ) -cuts of α_1 is denoted by $\alpha_{1(\lambda, \mu)}$ and defined by

$\alpha_{1(\lambda, \mu)} = \{x \in \frac{G}{J_{\alpha_1}^3(x)} > \lambda \text{ and } K_{\alpha_1}^3(x) < \mu\}$. Also it is a subgroup of G.

Definition-2.10: A fermatean fuzzy subgroup α_1 of G commutative if

$$J_{\alpha_1}^3(xh) = J_{\alpha_1}^3(hx) \text{ and}$$

$$K_{\alpha_1}^3(xh) = K_{\alpha_1}^3(hx) \text{ for all } x, h \in G.$$

However, a fermatean fuzzy subgroup α_1 is commutative if G is a commutative group.

Definition-2.11:

Let α_1 be a fermatean fuzzy subgroup of a fermatean fuzzy subgroup α_2 of G . Then α_1 is a normal in α_2 , denoted as $\alpha_1 < \alpha_2$ if

$$J_{\alpha_1}^3(xh) = J_{\alpha_1}^3(hx) \text{ and } K_{\alpha_1}^3(xh) = K_{\alpha_1}^3(hx) \Rightarrow J_{\alpha_1}^3(xhx^{-1}) = J_{\alpha_1}^3(h) \text{ and}$$

$$K_{\alpha_1}^3(xhx^{-1}) = K_{\alpha_1}^3(h) \text{ for all } x, h \in G.$$

Definition-2.12:

Assume G and H groups and $\delta: G \rightarrow H$ is a homomorphism. Suppose α_1 and α_2 are fermatean fuzzy subgroups of G and H respectively. Then δ induces a homomorphism from α_1 to α_2 that satisfies

$$(i) J_{\alpha_1}^3(\delta^{-1}(h_1h_2)) \geq \min\{J_{\alpha_1}^3(\delta^{-1}(h_1)), J_{\alpha_1}^3(\delta^{-1}(h_2))\} \text{ and}$$

$$K_{\alpha_1}^3(\delta^{-1}(h_1h_2)) \leq \max\{K_{\alpha_1}^3(\delta^{-1}(h_1)), K_{\alpha_1}^3(\delta^{-1}(h_2))\} \text{ for all } h_1, h_2 \in H.$$

$$(ii) J_{\alpha_2}^3(\delta(x_1x_2)) \geq \min\{J_{\alpha_2}^3(\delta(x_1)), J_{\alpha_2}^3(\delta(x_2))\} \text{ and}$$

$$K_{\alpha_2}^3(\delta(x_1x_2)) \leq \max\{K_{\alpha_2}^3(\delta(x_1)), K_{\alpha_2}^3(\delta(x_2))\} \text{ for all } x_1, x_2 \in G.$$

Main Results:

In the classical setting, we call a subgroup H of a group G is characteristic if $M^\delta = M$ for every automorphism, δ of G , where $M^\delta = \delta(M)$. Now, we establish the analogue of this idea as follows.

Definition-3.1: Let A be fermatean fuzzy subgroup of G , then A is characteristic (δ -invariant) if

$$J_{A^\delta}^3(x) = J_A^3(x),$$

$$K_{A^\delta}^3(x) = K_A^3(x), \text{ for all } x \in G \text{ for every automorphism } \delta \text{ of } G.$$

Hence $\delta(A) \subseteq A$ for every δ of $\delta \in \text{Aut}(G)$.

Definition-3.2: Let A be fermatean fuzzy subgroup in G and δ is a mapping of G . Define a fermatean fuzzy set A^δ in G by

$$J_{A^\delta}^3(x) = J_A^3(x^\delta),$$

$$K_{A^\delta}^3(x) = K_A^3(x^\delta), \text{ where } x^\delta = \delta(x) = x, \text{ for all } x \text{ in } G.$$

Proposition-3.3: Let A be fermatean fuzzy subgroup in G and $x \in G$. Suppose δ is an automorphism of G for $\delta(x) = gxg^{-1}$ for all $x \in G$, then $A^\delta = A$.

Proof: For A be fermatean fuzzy subgroup in G and $x \in G$, assume that $\delta: G \rightarrow G$ is characterized by $\delta(x) = gxg^{-1}$ for all $x \in G$.

$$\text{Then } J^3_{A^g}(x) = J^3_A(gxg^{-1}) = J^3_A(\delta(x)) = J^3_{A^\delta}(x),$$

$$K^3_{A^g}(x) = K^3_A(gxg^{-1}) = K^3_A(\delta(x)) = K^3_{A^\delta}(x).$$

Proposition-3.4: Let A be fermatean fuzzy subgroup in G and δ be an automorphism of G . Then A^δ is fermatean fuzzy subgroup in G .

Proof: For $x, g \in G$, we have

$$J^3_{A^\delta}(xg) = J^3_A((xg)^\delta) = J^3_A(x^\delta g^\delta) \text{ and}$$

$$K^3_{A^\delta}(xg) = K^3_A((xg)^\delta) = K^3_A(x^\delta g^\delta) \text{ because } \delta \text{ is homomorphism.}$$

Since A is a fermatean fuzzy subgroup of G , we have

$$J^3_A(x^\delta g^\delta) \geq \min\{J^3_A(x^\delta), J^3_A(g^\delta)\} = \min\{J^3_{A^\delta}(x), J^3_{A^\delta}(g)\} \text{ and}$$

$$K^3_A(x^\delta g^\delta) \leq \max\{K^3_A(x^\delta), K^3_A(g^\delta)\} = \max\{K^3_{A^\delta}(x), K^3_{A^\delta}(g)\}.$$

$$\text{Thus, } J^3_{A^\delta}(xg) \geq \min\{J^3_{A^\delta}(x), J^3_{A^\delta}(g)\} \text{ and } K^3_{A^\delta}(xg) \leq \max\{K^3_{A^\delta}(x), K^3_{A^\delta}(g)\}.$$

$$\text{In addition, } J^3_{A^\delta}(x^{-1}) = J^3_A((x^{-1})^\delta)$$

$$= J^3_A((x^\delta)^{-1})$$

$$= J^3_A(x^\delta) = J^3_{A^\delta}(x) \text{ and}$$

$$K^3_{A^\delta}(x^{-1}) = K^3_A((x^{-1})^\delta) = K^3_A((x^\delta)^{-1})$$

$$= K^3_A(x^\delta) = K^3_{A^\delta}(x).$$

Hence A^δ is fermatean fuzzy subgroup in G .

Theorem-3.5: Suppose $\delta: G \rightarrow G$ is an automorphism and A is a fuzzy subgroup in G . Then

A^δ is a fermatean fuzzy subgroup in G if and only if A is a fermatean fuzzy subgroup in G .

Proof: Suppose A is a fermatean fuzzy subgroup in G . Then using the procedure in the proposition-3.4, it is certain that A^δ is a fermatean fuzzy subgroup in G .

Conversely, if A^δ is a fermatean fuzzy subgroup in G . Then

$$J^3_{A^\delta}(xg) \geq \min\{J^3_{A^\delta}(x), J^3_{A^\delta}(g)\} \text{ and } J^3_{A^\delta}(x^{-1}) = J^3_{A^\delta}(x) \text{ and}$$

$$K^3_{A^\delta}(xg) \leq \max\{K^3_{A^\delta}(x), K^3_{A^\delta}(g)\} \text{ and } K^3_{A^\delta}(x^{-1}) = K^3_{A^\delta}(x), \forall x, g \in G.$$

$$\text{Thus, } J^3_{A^\delta}(xg) = J^3_A((xg)^\delta) = J^3_A(\delta(xg)) = J^3_A(xg)$$

$$\Rightarrow J^3_A(xg) \geq \min\{J^3_A(x), J^3_A(g)\}, \forall x, g \in G.$$

$$\begin{aligned}\text{In addition, } J^3_{A^\delta}(x^{-1}) &= J^3_A((x^{-1})^\delta) \\ &= J^3_A((x^\delta)^{-1}) \\ &= J^3_A((\delta(x))^{-1}) \\ &= J^3_A(x^{-1})\end{aligned}$$

$$\Rightarrow J^3_A(x^{-1}) = J^3_A(x) \text{ and}$$

$$\begin{aligned}K^3_{A^\delta}(x^{-1}) &= K^3_A((x^{-1})^\delta) \\ &= K^3_A((x^\delta)^{-1}) \\ &= K^3_A((\delta(x))^{-1}) \\ &= K^3_A(x^{-1})\end{aligned}$$

$$\Rightarrow K^3_A(x^{-1}) = K^3_A(x), \forall x \in G.$$

Hence A is a fermatean fuzzy subgroup of G .

Theorem-3.6: Every fermatean fuzzy characteristic subgroup if a fermatean fuzzy group is a fermatean fuzzy normal subgroup.

Proof: Let $x, g \in G$ and let A be fermatean fuzzy characteristic subgroup.

We need to show that $J^3_A(xg) = J^3_A(gx)$ and $K^3_A(xg) = K^3_A(gx), \forall x, g \in G$.

Let us assume that δ is the automorphism of G defined by $\delta(h) = x^{-1}g'h, \forall g' \in G$.

Now, since A is fermatean fuzzy characteristic subgroup we have $A^\delta = A$. Then

$$J^3_A(xg) = J^3_A(\delta(xg)) = J^3_A((xg)^\delta) = J^3_A(x^{-1}(xg)x) = J^3_A(gx) \text{ and}$$

$$K^3_A(xg) = K^3_A(\delta(xg)) = K^3_A((xg)^\delta) = K^3_A(x^{-1}(xg)x) = K^3_A(gx)$$

Hence A^δ is a normal subgroup.

Remark-3.7: Let A, B, C be fermatean fuzzy subgroups in G in which $C \subseteq B \subseteq A$

- (i) If C is a fermatean fuzzy characteristic subgroup of B and B is fermatean fuzzy characteristic subgroup of A , then C is a fermatean fuzzy characteristic subgroup of A .
- (ii) If C is a fermatean fuzzy normal subgroup of B and B is fermatean fuzzy normal subgroup of A , then C is a fermatean fuzzy normal subgroup of A .

Proposition-3.8: If B is fermatean fuzzy subgroup in G and A is fermatean fuzzy characteristic subgroup of B , then A is a characteristic subgroup of G . In addition, A is a

characteristic subgroup of G .

Proof: It is certain that A^* is a subgroup of G .

To explain that A^* is characteristic in G , it is sufficient to show that $\delta(A^*) \subseteq A, \forall \delta \in \text{Aut}(G)$.

Let $\delta \in \text{Aut}(G)$, then $J^3_{A^\delta}(x) = J^3_A(x)$ and $K^3_{A^\delta}(x) = K^3_A(x)$, since A is fermatean fuzzy characteristic subgroup of B .

Let $x \in A^*$, then $J^3_A(x) \geq 0$ and $K^3_A(x) \leq 0$, which gives

$$J^3_{A^\delta}(x) = J^3_A(\delta(x)) = J^3_A(x) \geq 0 \text{ and } K^3_{A^\delta}(x) = K^3_A(\delta(x)) = K^3_A(x) \leq 0.$$

So $\delta(x) \in A^*$. Hence $\delta(A^*) \subset A^*$ which concludes the result.

In addition, because A is a fermatean fuzzy characteristic subgroup of B and A^* is a characteristic subgroup of G , it shows A^* is a characteristic subgroup of B^* .

Theorem-3.9: Suppose G is finite and A is a fermatean fuzzy characteristic subgroup of B in G . Then $A_{[r,s]}$ and $A_{(r,s)}$ are characteristic subgroups of G .

Proof: It is obvious that, $A_{[r,s]}$ is a subgroup of G . Further we will clear that $A_{[r,s]}$ is characteristic in G by showing that $\delta(A_{[r,s]}) \subset A_{[r,s]}$, $\forall \delta \in \text{Aut}(G)$.

For $\delta \in \text{Aut}(G)$, we get $J^3_{A^\delta}(x) = J^3_A(x)$ and $K^3_{A^\delta}(x) = K^3_A(x)$, because A is a fermatean fuzzy characteristic subgroup.

Let $x \in A_{[r,s]}$. Then $J^3_A(x) \geq r$ and $K^3_A(x) \leq s$, so $\delta(x) \in A_{[r,s]}$. Thus $\delta(A_{[r,s]}) \subset A_{[r,s]}$. Hence, $A_{[r,s]}$ is a characteristic subgroup of G .

Similarly, $A_{(r,s)}$ is a characteristic subgroup of G .

Remark-3.10: Because A is a fermatean fuzzy characteristic subgroup of B , and $A_{[r,s]}$ and $A_{(r,s)}$ are characteristic subgroups of G , it follows that $A_{[r,s]}$ is a characteristic subgroup of $B_{[r,s]}$ and $A_{(r,s)}$ is a characteristic subgroup of $B_{(r,s)}$.

Theorem-3.11: Let A be a fermatean fuzzy subgroup of B in G , where G is finite. If $A_{[r,s]}$ and $A_{(r,s)}$ are characteristic subgroups of G , then A is fermatean fuzzy characteristic subgroup of B .

Proof: Since G is finite, $|A| < \infty$. Let $\text{Im}(A) = \{(r_0, s_0), (r_1, s_1), \dots, (r_n, s_n)\}$ with

$$r_0 > r_1 > \dots > r_n \text{ and } s_0 < s_1 < \dots < s_n.$$

By assumption, $A_{[r,s]} = \{x \in G \mid J^3_A(x) \geq r_j; K^3_A(x) \leq s_j\}$ is a characteristic subgroup of G , $\forall j = 0, 1, 2, \dots, n$.

Let $\delta \in \text{Aut}(G)$. Because $J^3_{A^\delta}(x) = J^3_A(\delta(x)) = J^3_{\delta^{-1}(A)}(x) = J^3_A(x)$ and

$$K^3_{A^\delta}(x) = K^3_A(\delta(x)) = K^3_{\delta^{-1}(A)}(x) = K^3_A(x), \text{ then } \text{Im}(A^\delta) = \text{Im}(A).$$

Moreover, $\forall p = 0, 1, 2, \dots, n$, we have $(A^\delta)_{[r_p, s_p]}$.

$$\text{Since } x \in (A^\delta)_{[r_p, s_p]} \Leftrightarrow J^3_{A^\delta}(x) \geq r_p, K^3_{A^\delta}(x) \leq s_p \Leftrightarrow J^3_A(\delta(x)) \geq r_p, K^3_A(\delta(x)) \leq s_p$$

$$\Leftrightarrow \delta(x) \in A_{[r_p, s_p]} \Leftrightarrow x \in \delta^{-1}(A_{[r_p, s_p]}) \Leftrightarrow x \in A_{[r_p, s_p]}.$$

Thus $A^\delta = A$ and hence A is fermatean fuzzy characteristic subgroup of B .

In what follows we validate that theorem-3.9 and theorem-3.11 are still true by discarding the finite order of G .

Theorem-3.12: Suppose A is fermatean fuzzy subgroup of B in G . Then the following statements are the same;

- (i) A is fermatean fuzzy characteristic subgroup of B .
- (ii) $A_{[r,s_p]}$ and $A_{(r,s_p)}$ are fermatean fuzzy characteristic subgroups in G .

Proof: In this proof (i) $\Rightarrow$ (ii)

Let $r, s \in \text{Im}(A)$, $\delta \in \text{Aut}(G)$ and $x \in A_{[r,s]}$.

Assume that A is fermatean fuzzy characteristic subgroup of B . Then

$$J_A^3(\delta(x)) = J_A^3(x) \geq r \text{ and } K_A^3(\delta(x)) = K_A^3(x) \leq s.$$

Thus $\delta(x) \in A_{[r,s]}$ and so, $\delta(A_{[r,s]}) \subseteq A_{[r,s]}$.

But $A_{[r,s]} \subseteq \delta(A_{[r,s]})$ by symmetry.

Let $x \in A_{[r,s]}$ and $g \in G$ where in $\delta(g) = x$.

$$\text{Then, } J_A^3(g) = J_A^3(\delta(g)) = J_A^3(x) \geq r \text{ and } K_A^3(g) = K_A^3(\delta(g)) = K_A^3(x) \leq s.$$

$$\Rightarrow g \in A_{[r,s]}, \text{ therefore } x \in \delta(A_{[r,s]}).$$

$$\text{Hence } A_{[r,s]} \subseteq \delta(A_{[r,s]}).$$

Thus, $A_{[r,s]}$ is a fermatean fuzzy characteristic subgroup in G .

Similarly, $A_{(r,s)}$ is a fermatean fuzzy characteristic subgroup in G .

Next, we prove (ii) $\Rightarrow$ (i).

$$\text{Let } x \in G, \delta \in \text{Aut}(G) \text{ and } J_A^3(x) = r, K_A^3(x) = s.$$

$$\text{Then } x \in A_{[r,s]} \text{ and } x \notin A_{[r^*,s^*]} \text{ for } r^*, s^* > r, s.$$

By hypothesis $\delta(A_{[r,s]}) = A_{[r,s]}$, so $\delta(x) \in A_{[r,s]}$ and hence

$$J_A^3(x) = J_A^3(\delta(x)) \geq r \text{ and } K_A^3(x) = K_A^3(\delta(x)) \leq s.$$

$$\text{Let } r^* = J_A^3(\delta(x)) \text{ and } s^* = K_A^3(\delta(x)).$$

$$\text{Assume } r^*, s^* > r, s, \text{ then } \delta(x) \in A_{[r^*,s^*]} = \delta(A_{[r^*,s^*]}).$$

Because δ is injective, we get $x \in A_{[r^*,s^*]}$, which is a contradiction.

$$\text{Thus, } J_A^3(\delta(x)) = r = J_A^3(x) \text{ and } K_A^3(\delta(x)) = s = K_A^3(x).$$

$$\Rightarrow A \text{ is fermatean fuzzy characteristic subgroup of } B.$$

Theorem-3.13: Let A be a fermatean fuzzy subgroup that is normal in a fermatean fuzzy subgroup B in G and assume δ is an automorphism of G which leaves an invariant subgroup A^* . Then δ induces an automorphism δ^* of B/A defined by $\delta^*(xA) = \delta(x)A, \forall x \in G$.

Proof: To start with, we confirm that δ^* is well defined.

Suppose $x, g \in G$ such that $xA = gA$. Then, it is sufficient to prove that $\delta(x)A = \delta(g)A$.

Since $xA = gA$, we get $J^3_{xA}(x) = J^3_{gA}(x)$ and $J^3_{xA}(g) = J^3_{gA}(g)$

$$\Rightarrow J^3_A(e) = J^3_A(g^{-1}x) \text{ and } J^3_A(x^{-1}g) = J^3_A(e)$$

$$\Rightarrow J^3_A(g^{-1}x) = J^3_A(x^{-1}g) = J^3_A(e) \Rightarrow g^{-1}x, x^{-1}g \in A^*.$$

Similarly, $K^3_{xA}(x) = K^3_{gA}(x)$ and $K^3_{xA}(g) = K^3_{gA}(g)$.

$$\Rightarrow K^3_A(e) = K^3_A(g^{-1}x) \text{ and } K^3(x^{-1}g) = K^3_A(e).$$

$$\Rightarrow K^3_A(g^{-1}x) = K^3(x^{-1}g) = K^3_A(e).$$

$$\Rightarrow g^{-1}x, x^{-1}g \in A^*.$$

By assumption, $\delta(A^*) = A^*$, then $\delta(g^{-1}x), \delta(x^{-1}g) \in A^*$.

Thus, $J^3_A(\delta(g^{-1}x)) = J^3_A(\delta(x^{-1}g))$ and $K^3_A(\delta(g^{-1}x)) = K^3_A(\delta(x^{-1}g))$.

Now, let $x \in G$, then

$$\begin{aligned} J^3_{\delta(x)A}(x^*) &= J^3_A(\delta(x^{-1})x^*) \\ &= J^3_A(\delta(x^{-1})\delta(g)\delta(g^{-1})x^*) \\ &\geq \min\{J^3_A(\delta(x^{-1})\delta(g)), J^3_A(\delta(g^{-1})x^*)\} \\ &= \min\{J^3_A(\delta(x^{-1}g)), J^3_A(\delta(g^{-1})x^*)\} \\ &= J^3_{\delta(g)A}(x^*) \text{ and } \end{aligned}$$

$$\begin{aligned} K^3_{\delta(x)A}(x^*) &= K^3_A(\delta(x^{-1})x^*) \\ &= K^3_A(\delta(x^{-1})\delta(g)\delta(g^{-1})x^*) \\ &\leq \max\{K^3_A(\delta(x^{-1})\delta(g)), K^3_A(\delta(g^{-1})x^*)\} \\ &= \max\{K^3_A(\delta(x^{-1}g)), K^3_A(\delta(g^{-1})x^*)\} \\ &= K^3_{\delta(g)A}(x^*). \end{aligned}$$

Thus, $J^3_{\delta(x)A}(x^*) \geq J^3_{\delta(g)A}(x^*)$ and $K^3_{\delta(x)A}(x^*) \leq K^3_{\delta(g)A}(x^*)$.

Similarly, $J^3_{\delta(g)A}(x^*) \geq J^3_{\delta(x)A}(x^*)$ and $K^3_{\delta(g)A}(x^*) \leq K^3_{\delta(x)A}(x^*)$.

Since $x^* \in G$ is arbitrary, then $J^3_{\delta(x)A}(x^*) = J^3_{\delta(g)A}(x^*)$ and $K^3_{\delta(x)A}(x^*) = K^3_{\delta(g)A}(x^*)$.

$$\Rightarrow \delta(x)A = \delta(g)A.$$

Thus, δ is well defined.

In addition, let $x, g \in G$. Because δ is homomorphism,

$$\delta(xg) = \delta(x)\delta(g) \text{ and so, } \delta(xg)A = \delta(x)A \delta(g)A.$$

$$\text{Hence } \delta^*(xgA) = \delta^*(x)A \delta^*(g)A \text{ and } \delta^*(xA) = \delta^*(x)A \delta^*(gA).$$

Therefore, δ^* is homomorphism.

Corollary-3.14: Using the theorem-3.13 is an automorphism if G is finite and δ is an automorphism.

Proof: Let $\delta(x) = \delta(g)$, for $x, g \in G$.

$$\text{Then } \delta(x)A = \delta(g)A, \text{ which gives } \delta^*(xA) = \delta^*(gA).$$

Hence $xA = gA$, since δ^* is injective.

$$\text{Thus, } J_{xA}^3(g) = J_{gA}^3(g).$$

$$\Rightarrow J_A^3(x^{-1}g) = J_A^3(e) \text{ and } K_A^3(g) = K_{gA}^3(e).$$

$$\Rightarrow J_A^3(x^{-1}g) = J_A^3(e).$$

$$\text{So, } x^{-1}g \in A^* \text{ and } x^{-1}g = e, \text{ since } A^* = \{e\}.$$

$$\text{Thus, } J_A^3(x) = J_A^3(g) \text{ and } K_A^3(x) = K_A^3(g) \Rightarrow x = g.$$

Hence δ is injective, which completes the proof.

Corollary-3.15: Let a fermatean fuzzy subgroup A be normal in a fermatean fuzzy subgroup B in G . If δ is an automorphism of G and $A^\delta = A$, then δ induces an automorphism of δ^* of B defined by $\delta^*(xA) = \delta(x)A$, for all $x \in G$.

Proof: Let $x, g \in G$. Then $xA = gA \Rightarrow xA^\delta = gA^\delta$.

$$\Leftrightarrow J_{xA^\delta}^3(x^*) = J_{gA^\delta}^3(x^*), K_{xA^\delta}^3(x^*) = K_{gA^\delta}^3(x^*), \forall x^* \in G.$$

$$\Rightarrow J_{A^\delta}^3(x^{-1}x^*) = J_{A^\delta}^3(g^{-1}x^*), K_{A^\delta}^3(x^{-1}x^*) = K_{A^\delta}^3(g^{-1}x^*), \forall x^* \in G.$$

$$\Rightarrow J_A^3(\delta(x^{-1}x^*)) = J_A^3(\delta(g^{-1}x^*)), K_A^3(\delta(x^{-1}x^*)) = K_A^3(\delta(g^{-1}x^*)), \forall x^* \in G.$$

$$\Rightarrow J_A^3(\delta(x^{-1})\delta(x^*)) = J_A^3(\delta(g^{-1})\delta(x^*)), K_A^3(\delta(x^{-1})\delta(x^*)) = K_A^3(\delta(g^{-1})\delta(x^*)),$$

$$\forall x^* \in G.$$

$$\Rightarrow J_{\delta(x)A}^3(\delta(x^*)) = J_{\delta(g)A}^3(\delta(x^*)), K_{\delta(x)A}^3(\delta(x^*)) = K_{\delta(g)A}^3(\delta(x^*)).$$

Thus, δ^* is well defined and injective. Certainly, δ^* maps B/A onto itself. Because δ^* is homomorphism, then the proof is completed by the similar wave length of the proof

theorem-3.13.

Theorem-3.16: Suppose $\delta: G \rightarrow G'$ is a homomorphism, A and B are fermatean fuzzy subgroups of G and G' respectively. Then $\delta(A) \subseteq B$ implies $J_B^3(\delta(x)) \geq J_A^3(x)$ and $K_B^3(\delta(x)) \leq K_A^3(x)$, for all $x \in G$.

Proof: Let $\delta(A) \subseteq B$ and $x \in G$.

Then, $J_A^3(\delta(x)) \geq J_{\delta(A)}^3(\delta(x))$

$$= \min\{J_A^3(g^*)/\delta(g^*) = \delta(x)\}$$

$$= J_A^3(x) \text{ and}$$

$$K_A^3(\delta(x)) \leq K_{\delta(A)}^3(\delta(x))$$

$$= \max\{K_A^3(g^*)/\delta(g^*) = \delta(x)\}$$

$$= K_A^3(x).$$

By inversely, let $J_B^3(\delta(x)) \geq J_A^3(x)$ and $K_B^3(\delta(x)) \geq K_A^3(x)$, for all $x \in G$.

Then, $J_{\delta(A)}^3(g) = \max\{J_A^3(x)/\delta(x) = g\}$

$$\leq \max\{J_B^3(\delta(x))/\delta(x) = g\}$$

$$= J_B^3(g) \text{ and}$$

$$K_{\delta(A)}^3(g) = \min\{K_A^3(x)/\delta(x) = g\}$$

$$\geq \min\{K_B^3(\delta(x))/\delta(x) = g\}$$

$$= K_B^3(g) \text{ for all } g \in G'.$$

Hence, $\delta(A) \subseteq B$.

CONCLUSION

In this article, the concepts of fermatean fuzzy characteristic maximal subgroups were shown. Various results concerning fermatean fuzzy characteristic subgroups were obtained and explained in detail.

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